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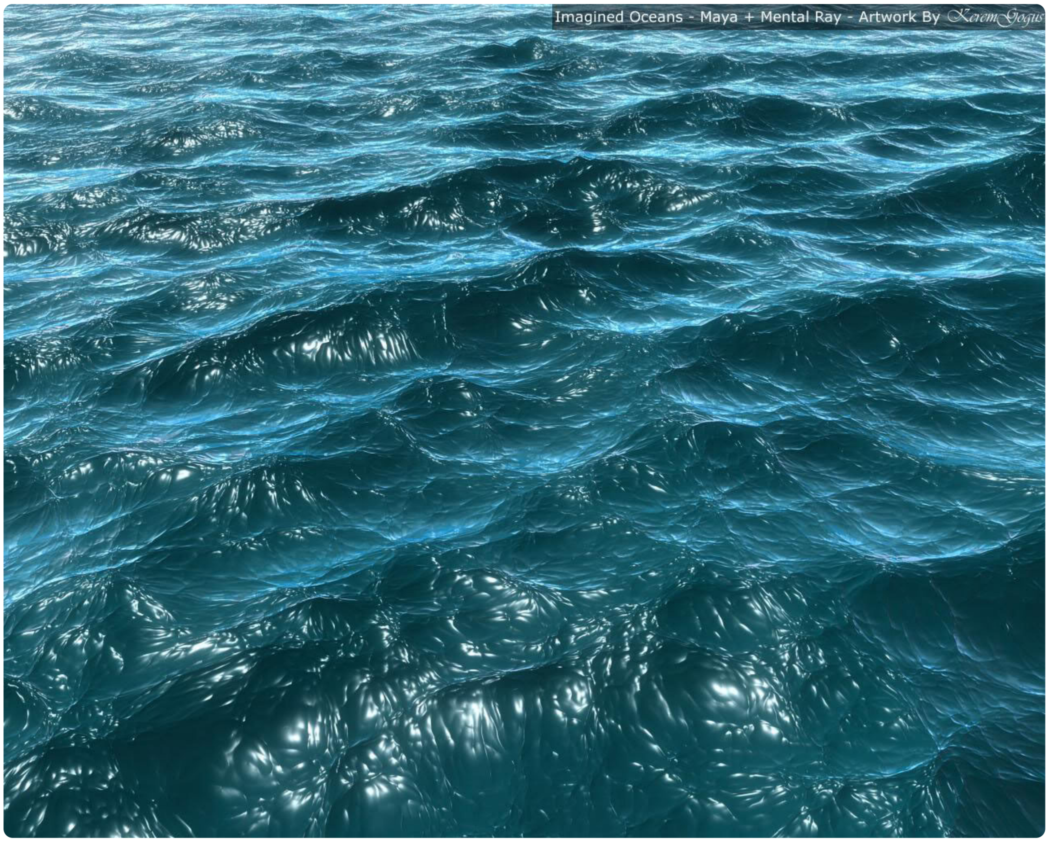
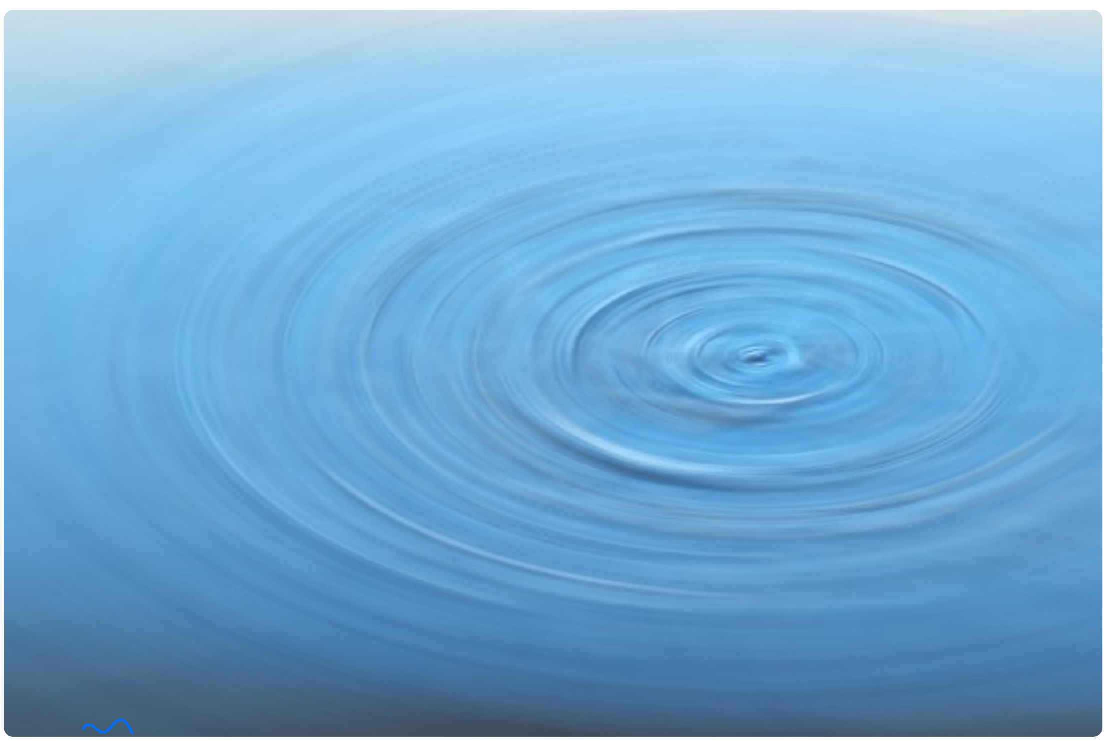

Imagined Oceans - Maya + Mental Ray - Artwork By Kerem Gogus



Lecture 2

Euler's equations (*)

Recall u = velocity vector field

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } \mathbb{R}^d$$

In $\mathbb{R}^2 \to$ additional auxilian functions

a) stream function $\psi : \mathbb{R}^2 \to \mathbb{R}$

b) velocity potential $\phi : \mathbb{R}^2 \to \mathbb{R}$

Conformal setting

Restrict to a domain $\Omega(t) \subset \mathbb{R}^2$, and try to understand (*)

(*) will change, because boundary conditions are needed!

Depending on the problem, we have the following types of bdd conds:

(a) impermeable bdd conds.

(b) free surface bdd. conds. (they are the hardest)

Types of flows that lead to water waves are inviscid, incompressible, irrotational flows.

fixed bdd $\to$ (a) $u \cdot n = 0$ on $\partial\Omega$ "no-flow" condition and states that no fluid goes through the bdd.

Obs: an inviscid fluid can slide: I do have a tangential component of the velocity vector field.

Obs: For Navier-Stokes.

$u = 0$ on $\partial\Omega$.

?. viscous fluid then, it "sticks" to the boundary.

Obs: If the boundary is not fixed, and moves with velocity V, then these bdd conds $\to$ Euler: $u \cdot n = V \cdot n$; N-S: $u = V$.

Obs: NS eq contain second-order spatial derivatives and therefore they require additional bdd conds.

$\Omega \subset \mathbb{R}^2$. IBVP for $u : \Omega \times \mathbb{R} \to \mathbb{R}^2$. velocity vector field.
$p : \Omega \times \mathbb{R} \to \mathbb{R}$. pressure.

$$(a) \quad \begin{cases} u_t + u \cdot \nabla u + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \\ u(0,x) = u_0 : \Omega \to \mathbb{R}^2 & \text{a given divergence free initial velocity.} \end{cases}$$

Note $p(t,x) + P(t)$ is not unique as one can absorb init funtions in t and still get a solution.

(b) On the free surface bdd. Def: A free surface is a bdd. whose location is not known apriori. $\Rightarrow$ this is very difficult to deal with.

Many free surface problems arise in fluid dynamics.

$\Rightarrow$ Stefan problem : melting ice problem.

$\to$ Hele-Shaw problem: for flow in a porus medium

$\to$ we will deal with free surface that comes where the water meets the air. $\Rightarrow$ water waves.

In water waves there are 2 free surface bdd conds

$\to$ kinematic boundary conditions $=$ free surface moves with the fluid

$\to$ dynamic boundary conditions $=$ balance of forces accros the interface

We work in 2D: The 2D water wave problems are different than 3D water waves in many ways and here is a short list:

— setting is different

— mathematical analysis (i.e. tools) are very different

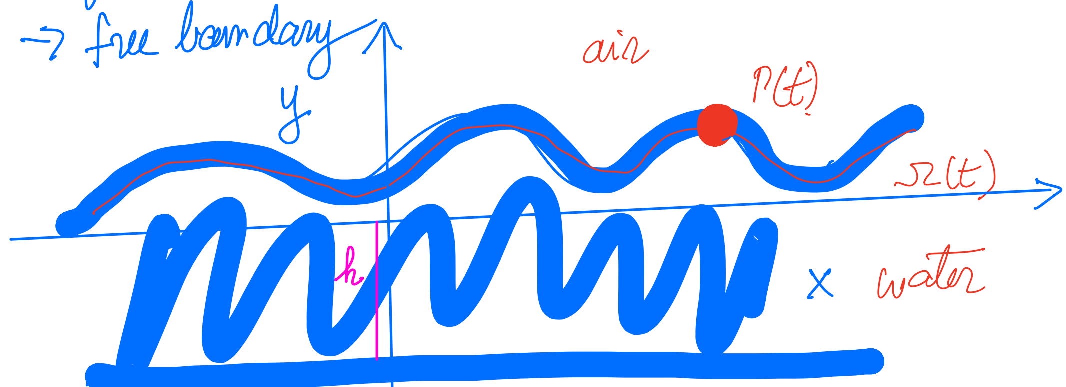
Common feature that the 2D and 3D water waves share is the large set of problem one might want to investigate

Setting: inviscid, incompressible flow (recast of incompressible Euler eqs)

→ gravity and/or surface tension

→ finite bottom (2D) $y = -h$, $h > 0$

→ free boundary



Comments:

- $\Omega(t)$ domain of the water.
- u describes the velocity of the fluid in this region.
- $\Gamma(t)$ is the interface.
- impose the corresponding bdd condition.
  - kinematic bdd conds (particles on $\Gamma(t)$ are staying on $\Gamma(t)$)

    $D_t = \partial_t + u \cdot \nabla$ is always tangent to $\Gamma(t)$

- dynamical bdd. conds

$$p = p_0 - \Gamma \mathbb{H}$$

p: water pressure; p_0: air pressure

→ Γ is the surface tension coeffient
→ $\mathbb{H}$ is the mean curvature on $\Gamma(t)$.

Now we want to go to water waves and here is where we need to take advantage and use the irrotationality. ($\omega = 0$, where $\omega = \nabla \times u$)

$\underline{\underline{\omega}}$ = vorticity = curl $u = \nabla \times u$.

$$\omega_t + u \cdot \nabla \omega = 0 \text{ in 2D (different evolution eq for } \omega \text{ in 3D)}$$

Now introduce the velocity potential $\phi : \Omega(t) \to \mathbb{R}$,

$u = \nabla \phi$, and using all given information

we have [i.e $(\nabla \cdot u = 0) + (\nabla \times u = 0)$] $\Rightarrow \Delta \phi = 0$ in $\Omega(t)$

- Velocity potential will helps to reduce the [problem] to the bdd. (hence reduce the space dimensions)
- Since all hypothesis are invariant with respect to rotation, is is not hard to see that

$$\frac{\partial \phi}{\partial n} = 0 \text{ on the bottom.}$$

n = normal unit vector to $\Gamma(t)$

Note: $\boxed{\text{KNOWING } \phi|_{\Gamma(t)} \text{ IMPLIES KNOWING } \phi|_{\Omega(t)}}$

Now we can restrict ourselves to $\Gamma(t)$ and hence getting closer to the water wave eqs.

The goal is to avoid tracking the entire fluid body and just focus on $\Gamma(t)$:

(i) one variable is the interface $\Gamma(t, x)$

(ii) the second variable is $\phi|_{\Gamma(t)}$

Knowing the solution to this system (assuming for now that we have one) should allow you to compute u at any given time t.

Our variables are:

- $y = \underline{\eta(t, \underline{x})}$, y represent the height
- $\psi(t, \underline{x}) := \phi|_{\Gamma(t)}$

$(\eta, \psi) \Rightarrow$ evolution described by them.

$\downarrow$

have a big role is played by the bold.

$\nearrow$

kinematic dynamic.

$\underline{u} = \underline{\nabla \phi}$

$\psi_0 = \phi|_{\Gamma(t_0)}$

give you this initial data. | this info is not enough: I need the $\dfrac{\partial \phi}{\partial n}$ of $\phi|_{\Gamma(t)}$

Dirichlet to Neumann operator = pseudodifferential operator of order 1

$$\begin{cases} \eta_t - G(\eta)\psi = 0 \\ \psi_t + g\eta - \sigma H(\eta) + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\dfrac{(\nabla\eta\,\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = 0 \end{cases}$$

$$G(\eta) = \sqrt{1+|\nabla\eta|^2}\,(\nabla\phi\cdot n).$$

$$H(\eta) = \nabla\cdot\left(\frac{\nabla\eta}{1+|\nabla\eta|^2}\right).$$

Zhakharov formulation

$$G(\eta) \approx |D| = H\circ D.$$

Next we will go over the derivation of Zhakharov formulation and move further to our final „setting"