Lecture 3

Euler eq: $\Omega(t) \subset \mathbb{R}^2$ + bdd conds

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p - g e_y \\ \nabla\cdot u = 0 \\ \nabla\times u = 0 \\ u_a = u(a, x) \\ D_t u \text{ is tangent to the free surface } \Gamma(t) \\ p = \underset{\substack{\| \\ P_{air}}}{0} - 2\sigma H \end{cases}$$

σ = surface tension coeff.

$$H = \operatorname{div}\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)$$

$$(\eta, \psi(t,x))$$

η: $y = \eta(t,x)$ ψ: $\phi\big|_{\Gamma(t)}$

Last time we wrote Zhakarov system:

$$(*) \begin{cases} \eta_t - G(\eta)\psi = 0 \\ \psi_t + g\eta - \sigma \underline{H(\eta)} + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\dfrac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = 0. \end{cases}$$

The kinematic bdd. cond: particle on the free surface, stays on the free surface.

$$D_t = \frac{D}{Dt} = \partial_t + u\cdot\nabla \text{ is tangent to the free surface at all times}$$

$$\underline{D_t(y - \eta(t,x))} = 0 \Longleftrightarrow$$

(x,y) Eulerian setting.

- location of a fixed fluid parcel $\to X(t) = (x(t), y(t))$ this is position at some time t.

- velocity of the fluid parcel

$$\underline{u}(x,t) = (u(t,x), v(t,x)) = (D_t x(t), D_t y(t))$$

Recall $D_t = \partial_t + (u,v)\cdot(\partial_x, \partial_y)$

$$= \partial_t + u\partial_x + v\partial_y.$$

Hence $D_t(y(t) - \eta(t,x)) = 0$

implies $v = D_t \eta$ $\to$ and we use $\to$ $\boxed{\underline{u} = \nabla\phi \quad (u,v) = (\phi_x, \phi_y)}$

$$\phi_y = D_t\eta \iff \eta_t + \underline{u}\cdot\nabla\eta = \phi_y$$

$$\eta_t + \nabla\phi\cdot\nabla\eta = \phi_y$$

$\circledast$ $\boxed{\eta_t + \phi_x\eta_x = \phi_y \text{ on } \eta(t,x) = y}$

this is the kinematic bdd. cond

The dynamic boundary condition

On the free surface $y = \eta(t,x)$ we require the continuity of the pressure field p: $\boxed{p = p_{air} - \sigma H}$, (A)

Now we couple this boundary condition to the momentum equation

$$u_t + u\cdot\nabla u + \nabla p + g\,e_y = 0, \quad \text{in } \Omega(t)$$

where $e_y = (0,1)$ unit vector

To do so we recall $u = \nabla\phi$, where ϕ is the velocity potential function that solves a Laplace eq in $\Omega(t)$ + bdd cond. Plug in $u = \nabla\phi$ in the momentum-eq and integrate in x and y

$$\iint_{\mathbb{R}\,-\infty}^{\eta(t,x)} \nabla\phi_t + \nabla\phi\cdot\nabla(\nabla\phi) + \nabla p + g\,e_y \,dx\,dy = \iint_{\mathbb{R}\,-\infty}^{\eta(t,x)-y} 0\,dx\,dy.$$

Integrate to get

$$\boxed{\phi_t + \frac{1}{2}|\nabla\phi|^2 + p - g\eta = 0} \quad ②$$

Bernoulli's equation follows

we have zero because any constant ($F(t)$ form) will be included in p

Plug in eq ① in ② gives

$$\boxed{\phi_t + \frac{1}{2}|\nabla\phi|^2 - g\eta = \Gamma.H}$$

Still we are not at Zhakharov's formulation but we will work towards that.

Recall $G(\eta)\psi = \sqrt{1+\eta_x^2}\;\nabla\phi\cdot n$ and use this towards Zhakarov's formulation

Now the normal to the surface given by $\eta(t,x) = y$, $F(x,y,t) = y - \eta(t,x)$ is given by

$$n_\eta = \frac{\nabla F}{|\nabla F|} = \frac{(-\eta_x, 1)}{\sqrt{1+\eta_x^2}} \quad \text{(unit normal to the surface pointing upwards)}$$

Then the velocity normal to the surface is given by

$$\nabla\phi\cdot n_\eta = \frac{-\phi_x\eta_x + \phi_y}{\sqrt{1+\eta_x^2}}.$$

Rewrite this relation $\Rightarrow$ $\sqrt{1+\eta_x^2}\;\nabla\phi\cdot n_\eta = -\phi_x\eta_x + \phi_y$. From eq ⊛ we can replace ϕ_y and get

$$\underbrace{\sqrt{1+\eta_x^2}\;\nabla\phi\cdot n_\eta}_{G(\eta)\psi} = \cancel{-\phi_x\eta_x} + \eta_t + \cancel{\phi_x\eta_x}$$

$$= \eta_t$$

This now reads as $\eta_t = G(\eta)\Psi$ which is the first eq we aimed for.

Now to get the second eq we need to use Bernoulli's eq and restrict it further to $\Gamma(t)$. The issue here is that $|\nabla\phi|^2$ needs to be expressed in terms of Ψ. and for this we need to solve the following system

$$\begin{cases} G(\eta)\Psi = \sqrt{1+\eta_x^2}\,\nabla\phi\cdot n = -\phi_x\eta_x + \phi_y \\ \Psi_x = \nabla\phi\,(1,\eta_x) = \phi_x + \phi_y\eta_x \end{cases}$$

and solve for ϕ_x and ϕ_y as functions of Ψ, and you arrive at the second eq

---

Hence the Zhakarov's system is nonlinear, nonlocal but 1D in our case. Now, we write down some conserved quantities

① $$\mathcal{H} = \frac{g}{2}\int_{\mathbb{R}} \eta^2\,dx + \sigma\int_{\mathbb{R}} \left(\sqrt{1+\eta_x^2} - 1\right)dx + \frac{1}{2}\iint_{\Omega(t)} |\nabla_{(x,y)}\phi|^2\,dx\,dy$$

Energy, and it is conserved along the flow!

② Conservation of the horizontal momentum

$$\mathcal{M} = \int_{\mathbb{R}} \eta\Psi_x\,dx \to (\text{invariance in space translations})$$

Noether's theorem: for every symmetry you can associate a conserved quantity.

- Scaling law $\to$ $g>0$ $(\sigma=0)$ $\begin{cases} \Psi_\lambda(t,x) = \lambda^{-3/2}\Psi(\sqrt{\lambda}t, \lambda x) \\ \eta_\lambda(t,x) = \lambda^{-1}\eta(\sqrt{\lambda}t, \lambda x) \end{cases}$

$\eta \in \dot{H}^{3/2}(\mathbb{R})$, $\Psi \in \dot{H}^2(\mathbb{R})$

$\downarrow g=0 \ (\sigma>0)$ $\begin{cases} \psi_\lambda(t,x)=\lambda^{-\frac{1}{2}}\psi(\lambda^{3/2}t,\lambda x) \\ \eta_\lambda(t,x)=\lambda^{-1}\eta(\lambda^{3/2}t,\lambda x) \end{cases}$

$\eta \in \dot{H}^{3/2}(\mathbb{R})$

$\psi \in \dot{H}^{1}(\mathbb{R})$

- σ, g, h = depth, $\omega=0$! ω = constant

Potential problems are

1) $h=\infty$, g, $\sigma=0$, $\omega=0$ gravity water waves
2) $\sigma>0$, $g=0$, $h=\infty$, $\omega=0$ capillary water wave.
3) h = finite, g, $\omega=0$

$G(\eta)\psi$ = we do not want to work with this operator, we want to diagonalize it meaning we would like to get something like $G(\eta)\psi \approx |D| = H \circ D$.

Linearization = around zero solution.

$(\eta,\psi)=0 \to \begin{cases} \eta \text{ is flat.} \\ \text{velocity } \psi=0 \end{cases}$

The linearization is dispersive which implies that the nonlinear problem is dispersive.

$$G(\eta)\psi = |D| = H\circ D$$

$\begin{cases} \eta_t - |D|\psi = 0 \\ \psi_t - \sigma\eta_{xx} + g\eta = 0 \end{cases}$ $\to$ here we discarded the quadratic and higher order terms in the linearization

$$\boxed{\eta_{tt} = -(\sigma|D|^3 + |D|)\eta} \quad ** \qquad H \cdot H = -I.$$

(τ, ξ)

$\tau^2 = \sigma|\xi|^3 + g|\xi|$: principle symbol associated to the equation **

<u>Discussion of the symbol.</u>

- the roots real. real. → eq is dispersive time. because the ξ part in the symbol is not linear.
- if the roots are not real → ill posed. problem
- $\sigma \geq 0$
- $\sigma = 0$, g (sing becomes important!)
- $\sigma = 0 = g$.

We assume $\sigma \geq 0$ either $\sigma = 0$, $g \geq 0$, and this assumption remains valid for this course

<u>Normal derivative of the pressure</u>

$\frac{\partial p}{\partial n}$ its sign matters! i.e. $-\frac{\partial p}{\partial n} > 0$. → this is our convention

↳ <u>Taylor sign condition</u>

||

in the linearization around $(0,0) = (\eta, \psi)$, we get that

$\underline{\sim g|\xi|}$, $\frac{\partial p}{\partial n} = g\eta$