Lecture 4

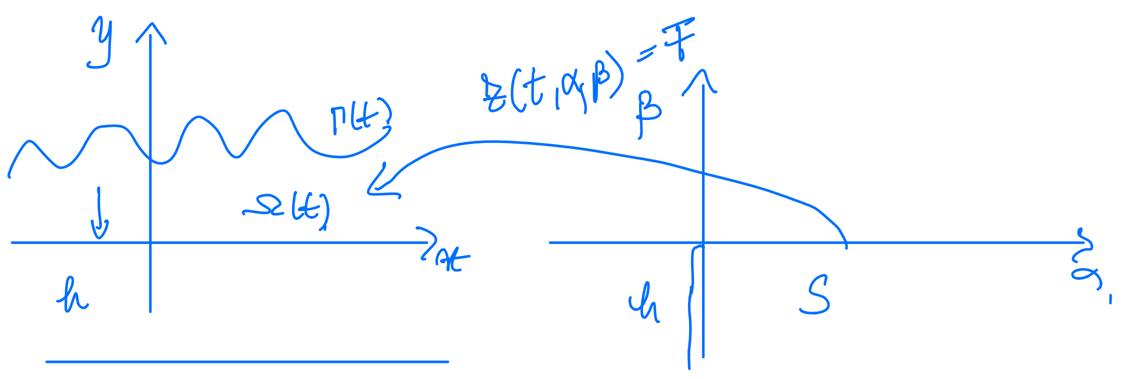
= Holomorphic coordinates =

- first time using in this coordinates Ovsiannikov in 1960.
- independently Wu, Dyachenko-Zakharov and Kuznetsov
- they are the first to compute the 2-d. water waves in conf coordinates (but their eq are different)
    - a) Wu wrote the eq like a second order evolution in time.
    - b) DZK → the eq are similar ... but different. since their goal is different.

The choice of coordinates restricts us to 2D.

! 3D Wu has an adaptation of what conformal coordinate can be envisioned (via Clifford algebras)

Note



Historically, the conformal coordinate are 100 years old and they were used to study special sols to water waves (eg. traveling waves)...

$G(\eta) \approx \underline{H \circ D} = |D|$

diagonalize G

Conformal Eq for Water Waves

$$H[f](\alpha) = \frac{1}{\pi}\, p.v. \int_{-\infty}^{+\infty} \frac{f(\alpha')}{\alpha-\alpha'}\, d\alpha' \Big/ H[e^{i\xi\alpha}] = -i\,\mathrm{sgn}\,\xi\, e^{i\xi\alpha}$$

where $f: \mathbb{R} \to \mathbb{C}$

> The following way of writing the eq is joint work with John Hunter, Daniel Tataru, Ifrim

$P = \frac{1}{2}(I - iH)$ = projections onto the boundary values of functions that are holomorphic in the lower half plane and satisfy suitable condition on the boundary.

neg freq

$\bar{P}$ = is the conjugation of P

$P[f] = f. \Rightarrow$ holomorphic functions

$P[\bar{f}] = 0 \Rightarrow$ antiholomorphic function. (antiholomorphic functions are projected to zero by P)

$\bar{P}[\bar{f}] = \bar{f}$. Also $P + \bar{P} = I$ (easy to check)

If $f + ig$ is holomorphic in the lower half plane (denoted by $\mathbb{H}$)

$$\mathbb{H} = \{\alpha + i\beta : \beta < 0\}.$$

The boundary value of $f+ig$ is $\underline{F + iG}$, on the real axes parametrized here by α.

Then the imaginary and real parts are connected via the H transform.

$$F = HG, \text{ and implicitly, using } H^2 = -I \Rightarrow -HF = +G$$

Property of Hilbert transf: $H[ab - Ha\,Hb] = aHb + bHa$

Proof: $(a+ib)^2 = a^2 + 2abi - b^2$

Here we also have $a = Hb$ So, $a^2 - b^2 = H[2ab]$

So $Ha = -b$ (apply H to ↑) $H[a^2 - b^2] = -2ab = 2aHa$ which is the formula above with $b = a$.

$$\begin{cases} f_\alpha = g_\beta \\ f_\beta = -g_\beta. \end{cases} \rightarrow \quad f_\beta\Big|_{\beta=0} = -G_\alpha = HF_\alpha$$

$$g_\beta\Big|_{\beta=0} = F_\alpha = HG_\alpha.$$

$$f\Big|_{\beta=0} = F$$

$$g\Big|_{\beta=0} = G.$$

Conformal map : $\mathcal{F}(t) : \mathbb{H} \to \Omega(t)$

$$\begin{cases} x = x(\alpha, \beta, t) \\ y = y(\alpha, \beta, t) \end{cases}$$

$$\underline{\mathcal{F}(\alpha, \beta) = (x(\alpha, \beta, t), y(\alpha, \beta, t))}$$

$$\begin{cases} x_\alpha = y_\beta \\ x_\beta = -y_\alpha \end{cases}$$ the Cauchy-Riemann's equations

$$\begin{cases} \partial_\alpha = x_\alpha \partial_x + y_\alpha \partial_y \\ \partial_\beta = x_\beta \partial_x + y_\beta \partial_y \end{cases}, \quad \begin{cases} \partial_x = \frac{1}{j}(x_\alpha \partial_\alpha + x_\beta \partial_\beta) \\ \partial_y = \frac{1}{j}(y_\alpha \partial_\alpha + y_\beta \partial_\beta) \end{cases}$$

where $j = x_\alpha y_\beta - x_\beta y_\alpha =$

$$= x_\alpha^2 + y_\alpha^2.$$

We assume. (we impose it)

$$x(\alpha, \beta, t) + i y(\alpha, \beta, t) \sim \alpha + i\beta \qquad \alpha, \beta \to \infty$$

which means $\mathcal{F}$ approaches the identity map at infinity.

If $\phi : \Omega(t) \to \mathbb{R}$, $\Psi = \phi \circ \mathcal{F} : \mathbb{H} \to \mathbb{R}$.

$$\Psi(\alpha, \beta, t) = \phi(x(\alpha, \beta, t), y(\alpha, \beta, t), t)$$

$$\psi_{\alpha\alpha} + \psi_{\beta\beta} = j\,(\phi_{xx} + \phi_{yy}).$$

velocity potential which we know it harmonic, $\psi(\alpha,\beta,t)$ is also harmonic.

Then velocity component $(u,v) = (\phi_x, \phi_y) \to$ in terms of this conformal velocity potential ψ.

$$u = \frac{1}{j}\,(x_\alpha \psi_\alpha + x_\beta \psi_\beta)$$

$$v = \frac{1}{j}\,(y_\alpha \psi_\alpha + y_\beta \psi_\beta)$$

$$\Rightarrow \quad u^2 + v^2 = \frac{1}{j}\,(\psi_\alpha^2 + \psi_\beta^2)$$

- $f : \Omega(t) \to \mathbb{C}$

  $g = f \circ \mathcal{F} : \mathbb{H} \to \mathbb{C}$

  $g(\alpha,\beta,t) = f(x(\alpha,\beta,t), y(\alpha,\beta,t), t)$

  (3). $\begin{cases} g_t = f_t + x_t f_x + y_t f_y \\ f_t = g_t - \frac{1}{j}(x_\alpha x_t + y_\alpha y_t) g_\alpha - \frac{1}{j}(x_\beta x_t + y_\beta y_t) g_\beta \end{cases}$

<u>Boundary values</u>

$x(\alpha,\beta,t)$

$\left.\begin{aligned} x(\alpha,a,t) &= X(\alpha,t) \\ y(\alpha,a,t) &= Y(\alpha,t) \end{aligned}\right\}$ boundary $\Gamma(t)$ of $\Omega(t)$

Since, $(x-\alpha) + i(y-\beta)$ holomorphic map in lower half plane and decay to infinity.

Hence, on the boundary the real and imaginary parts are connected via $\begin{cases} Y = -H(X-\alpha) \\ X = \alpha + HY. \end{cases}$

$\Psi(\alpha,t) := \tau(\alpha,0,t)$. $\to$ boundary values of the conformal velocity potential.
$\uparrow$ this is a notation

and define $\Theta(\alpha,\beta)$

$$\Psi(\alpha,t) = \underset{\uparrow \text{ Hilbert trans}}{H}\,\Theta \quad , \quad \Theta = -H\Psi$$

Since ψ is harmonic.

$$\psi_\beta\big|_{\beta=0} = H\Psi_\alpha = -\Theta_\alpha$$

Kinematic BC.

A normal to the boundary Γ is $(-Y_\alpha, X_\alpha)$

The kinematic bdd conds states

the normal component of the velocity of the boundary = normal component of the fluid velocity

$$(X_t, Y_t)\cdot(-Y_\alpha, X_\alpha) = (u,v)\cdot(-Y_\alpha, X_\alpha) \text{ on } \Gamma(t)$$

Now $\boxed{X_\alpha Y_t - Y_\alpha X_t = \psi_\beta\big|_{\beta=0} = -\Theta_\alpha}$ (1)

$\boxed{\begin{matrix} X_t \\ Y_t \end{matrix}}$ eqs!

kinematic bdd cond in conformal coordinates

I need an additional eq for x_t, y_t. and for this we need to use any additional info we have.

$c = \alpha + i\beta$

$z(c) = \underline{x(\alpha,\beta)} + i y(\alpha,\beta)$

$z' = x_\alpha + i y_\alpha$. is nonzero in the lower half plane.

$z' \to 1 \quad c \to \infty$

$w(c) = \frac{1}{z'(c)} - 1$ is holomorphic and decays.

$= \frac{1}{x_\alpha + i y_\alpha} - 1$

$W := w\big|_{\beta=0}$ on the real axis

$$W = \underline{\left(\frac{x_\alpha}{j} - 1\right)} - i\,\underline{\frac{y_\alpha}{j}}$$

$j = j\big|_{\beta=0}$

$j = x_\alpha^2 + y_\alpha^2$

$j = x_\alpha^2 + y_\alpha^2$.

$$\frac{x_\alpha}{j} - 1 = -H\left[\frac{y_\alpha}{j}\right]$$

Divide the Riemann bdd cond by j, $\underline{x_t = H y_t}$, the Hilbert transf identity.

$$(2) \quad \boxed{x_\alpha x_t + y_\alpha y_t = -j H\left[\frac{\theta_\alpha}{j}\right]}$$

(1) + (2) $\to x_t, y_t =$

$$\begin{cases} x_t = -H\left[\frac{\theta_\alpha}{j}\right] x_\alpha + \frac{\theta_\alpha}{j} y_\alpha \\ y_t = -\frac{\theta_\alpha}{j} x_\alpha - H\left[\frac{\theta_\alpha}{j}\right] y_\alpha. \end{cases}$$

Dynamic bdd cond

$$(3) \quad \boxed{\varphi_t + H\left[\frac{\theta_\alpha}{j}\right]\varphi_\alpha + \frac{1}{2j}\left(\varphi_\alpha^2 - \theta_\alpha^2\right) + y = 0}$$

Complex form:

$$Z := X + iY$$
$$Q := \Psi + i\Theta$$
$$F = H\left[\frac{\Theta_\alpha}{J}\right] + i\frac{\Theta_\alpha}{J}$$

boundary values of functions that are holomorphic in lower half plane $J = |Z_\alpha|^2$

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right]$$

$$!\begin{cases} Z_t + F Z_\alpha = 0 & \to \text{kinematic bdd cond} \\ Q_t - i(Z-\alpha) + F Q_\alpha + P\left[\frac{|Q_\alpha|^2}{J}\right] = 0 & \to \text{dyn + Bernoulli eq} \\ & \quad \text{bdd} \end{cases}$$

$\to F =$

We have one degree of freedom in the choice of α, namely the the horizontal translations $\lim_{|\alpha|\to\infty} Z(\alpha) - \alpha = 0$

$\to W := Z - \alpha$

W notation

$$\text{WW eqs}\ \begin{cases} W_t + F(1+W_\alpha) = 0 \\ Q_t + FQ_\alpha - igW + P\left[\frac{|Q_\alpha|^2}{J}\right] = 0 \end{cases}$$

$W \to$ free surface
$Q \to$ complex velocity potential

$$J = |Z_\alpha|^2 = |1 + W_\alpha|^2.$$

Space where this evolution happens is the space of holomorphic functions, admit the bounded extensions in the lower half plane, (i.e functions that have their Fourier support in $(-\infty, 0]$

infinite depth, $\sigma = 0$, g, (the set of eqs above are for the infinite gravity (no surface tension)