Lecture 5 – discussion needed before LWP theory

Recall that the system we derived for the 2D water waves (gravity, no surface tension, infinite depth) in conformal coordinates

free bdd → complex velocity potential →

$$\begin{cases} Z_t + F Z_\alpha = 0 \\ Q_t + F Q_\alpha - i(Z-\alpha) + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0 \end{cases} , \alpha \in \mathbb{R}$$

$J = |Z_\alpha|^2$ → jacobian of the change of coordinates

$F = P\left[\dfrac{Q_\alpha - \bar{Q}_\alpha}{J}\right]$

$W := Z - \alpha$ → based on the condition we impose at infinity (flatness of the surface at ∞)

$$(WW)\quad \begin{cases} W_t + F(1+W_\alpha) = 0 \\ Q_t + FQ_\alpha - iW + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0 \end{cases}, \quad J = |1+W_\alpha|^2$$

Observations on our system:

1. Nonlinear system
2. Nonlocal because $P = \frac{1}{2}(I - iH)$ H = Hilbert transf
3. Space: „space of holomorphic functions" – functions that are restrictions of bounded holom functions defined in the whole lower half plane = functions that have Fourier transform supported $[-\infty, 0]$

$\boxed{\text{abuse of terminology}}$

On $\mathbb{R}$ → $P(f) = f$ (the space is fully determined by this relation)

On $\mathbb{T}$ (torus) → similar formulation of this problem, but with small adjustments ("constants" are not „seen" by Hilbert trans)

4. all the functions in (WW) are holomorphic.

5. observe F is like a transport term but unfortunately is complexed valued; We "extract" the real part of F, we call this quantity b:

$$b := P\left[\frac{Q_\alpha}{J}\right] + \bar{P}\left[\frac{\bar{Q}_\alpha}{J}\right]$$

Next step, I differentiate with respect to $\alpha \Rightarrow$ it becomes a quasilinear problem.

We get a system in (W_α, Q_α) which is self-contained in this variables, which should trigger some hope:

$$\begin{cases} W_{\alpha t} + b W_{\alpha\alpha} + \dfrac{1}{1+\bar{W}_\alpha}\left(Q_{\alpha\alpha} - \dfrac{Q_\alpha}{1+W_\alpha} W_{\alpha\alpha}\right) + (1+W_\alpha)\bar{F}_\alpha \\ \qquad - \left[\dfrac{\bar{Q}_\alpha}{1+\bar{W}_\alpha}\right]_\alpha = 0 \\ Q_{t\alpha} + b Q_{\alpha\alpha} - i W_\alpha + \dfrac{1}{1+\bar{W}_\alpha}\dfrac{Q_\alpha}{1+W_\alpha}\left(R_{\alpha\alpha} - \dfrac{Q_\alpha}{1+W_\alpha} W_{\alpha\alpha}\right) \\ \qquad + Q_\alpha \bar{F}_\alpha + \bar{P}\left[\dfrac{|Q_\alpha|^2}{J}\right]_\alpha = 0 \end{cases}$$

quasilinear

How we do not really know what the unknowns / versus the coefficient.

- we do not know principal part / perturbation part.

<u>Linearization</u>: The linearization around a solution measures infinitesimals deformation of solutions when you change the initial data. Assuming we have a one parameter family of solutions, smoothly depending on the parameter, m,

$(W(m), Q(m))$ then the linearized variables

$\left(\frac{d}{dm} W(m) \Big|_{m=0}, \frac{d}{dm} Q(m) \Big|_{m=0} \right)$. We obtain the linearization by differentiating the given eq (are WW) with respect to the parameter m.

$$\frac{d}{dm} W(m) \Big|_{m=0} = w, \qquad \frac{d}{dm} Q(m) \Big|_{m=0} = q$$

Obs: Because of the translation invariance (spatial + time) symmetry we here (W_α, Q_α) are sols of the linearized eqs

(W_t, Q_t) are sols of the linearized eqs.

The linearization is given by

$$\begin{cases} w_t + b w_\alpha + \frac{1}{1+\overline{W}_\alpha}\left(q - \frac{Q_\alpha}{1+W_\alpha} w_\alpha\right) = \text{lower order terms in } (\bar{w}, \bar{q}) \\ q_t + b q_\alpha - i g w + \frac{1}{1+\overline{W}_\alpha} \frac{Q_\alpha}{1+W_\alpha}\left(q_\alpha - \frac{Q_\alpha}{1+W_\alpha} w_\alpha\right) = \text{lower order terms in } (\bar{w}, \bar{q}) \end{cases}$$

eg. $w_t = P[R\, \bar{w}_\alpha] + \ldots \quad w \in L^2$

$\|R_\alpha\|_{L^\infty} \|w\|_{L^2}$

Decide if this problem is hyperbolic or not:

$$\begin{pmatrix} w \\ z \end{pmatrix}_t = A \begin{pmatrix} w_\alpha \\ z_\alpha \end{pmatrix} + \begin{pmatrix} ?\,w \\ ?\,z \end{pmatrix} + \dots$$

A matrix $\to$ eigenvalues $\to$ real hyperbolic system
$\to$ not real then is not hyperbolic.

$$A = \begin{pmatrix} b - \dfrac{Q_\alpha}{1+W_\alpha}\,\dfrac{1}{1+\bar{W}_\alpha} & \dfrac{1}{1+\bar{W}_\alpha} \\ -\dfrac{1}{1+\bar{W}_\alpha}\left(\dfrac{Q_\alpha}{1+W_\alpha}\right)^2 & b + \dfrac{Q_\alpha}{1+W_\alpha}\cdot\dfrac{1}{1+\bar{W}_\alpha} \end{pmatrix}$$

$$\det(A - \lambda I) = 0 \iff (b-\lambda)^2 = 0 \qquad \lambda_{1,2} = b \text{ (real number)}.$$

Diagonalize the system:

$$(A - \lambda I)\cdot X = 0 \xrightarrow[\text{corresp to } \lambda = b]{\text{eigenvector}} \begin{pmatrix} 1 \\ \dfrac{Q_\alpha}{1+W_\alpha} \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 \\ \dfrac{Q_\alpha}{1+W_\alpha} & 1 \end{pmatrix} \to P^{-1} = \begin{pmatrix} 1 & 0 \\ -\dfrac{Q_\alpha}{1+W_\alpha} & 1 \end{pmatrix}$$

$$P^{-1} A P = \begin{pmatrix} b & \dfrac{1+W_\alpha}{1+\bar{W}_\alpha} \\ 0 & b \end{pmatrix}$$

The diagonalization operator

$$A(w,z) = \Big(w,\ w - \underbrace{\frac{Q_\alpha}{1+W_\alpha}}_{= R} z\Big) = (w,\ w - Rz)$$

R = velocity of fluid restricted to the boundary.

The diagonalized linearized eqs

$$\begin{cases} (\partial_t + b\partial_\alpha) w + \frac{1}{1+\overline{W}_\alpha} r_\alpha + \frac{R_{\alpha\alpha}}{1+\overline{W}_\alpha} w = \text{--- lower order term} \\ (\partial_t + b\partial_\alpha) r - i \frac{g+a}{1+W_\alpha} w = \text{---} \end{cases}$$

Comment: $\begin{cases} w_t + r_\alpha = 0 \\ r_t - igw = 0 \end{cases}$ $\rightarrow$ is a linear well-posed evolution in the space.

$$\dot{\mathcal{H}}_0 := L^2 \times \dot{H}^{\frac{1}{2}}$$

$$E_0(w,r) = \int \frac{g}{2} |w|^2 + \frac{1}{2i}(r\bar{r}_\alpha - \bar{r} r_\alpha)\, d\alpha.$$

$\tau = \pm\sqrt{g|\xi|}$ $\rightarrow$ dispersive $\rightarrow$ original problem is dispersive

For the full linearized eqs we have the following energy

$$E^{(2)}_{lin}(w,r) = \int_{\mathbb{R}} \underbrace{(g+a)}_{>0}|w|^2 + \Im(r\bar{r}_\alpha)\, d\alpha. \rightarrow \text{next time.}$$

$$a := i\left(\bar{P}[\bar{R}R_\alpha] - P[R\bar{R}_\alpha]\right)$$

$$g+a = -\frac{\partial p}{\partial n} \geqslant 0$$

Next return to (W_α, R_α) system (and use your findings from the linearized eq and $(W_\alpha, R = \frac{Q_\alpha}{1+W_\alpha})$)

$$\begin{cases} (W_\alpha)_t + b(W_\alpha)_\alpha + \frac{(1+W_\alpha)}{1+\overline{W}_\alpha} R_\alpha = \text{---} \\ R_t + bR_\alpha - i\left(\frac{W_\alpha - a}{1+W_\alpha}\right) = 0 \end{cases}$$

this is the diagonalized differentiated system in (W_α, R)