Summer school lecture 1

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1. Notations

Space $\mathbb{R}^d$, $(x_1, \dots x_d)$

space-time $\mathbb{R} \times \mathbb{R}^d$, $(t, x_1, \dots x_d)$

Derivatives $\partial_j = \dfrac{\partial}{\partial x_j}$, $\partial_t = \dfrac{\partial}{\partial t}$

2. PDE's.

$|\alpha| \leq K$

$$F(u, \partial u, \dots \partial^\alpha u) = 0$$

$$\partial^\alpha u = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} u$$

order of the PDE.

$$\alpha = (\alpha_1, \dots, \alpha_d)$$

$$|\alpha| = |\alpha_1| + \dots |\alpha_d|$$

- Linear pde's : F is linear
- Semilinear pde's: F is linear with constant coeff. in highest deriv. $|\alpha| = K$

Example: $\Delta u = u^3$

- quasilinear pde's: F is linear in highest deriv., but not with constant coeff.

  Example:

$$\partial_j a^{jk}(u)\,\partial_j u = 0$$

- fully nonlinear pde's:

  F is nonlinear to highest order

$$\det |\partial^2 u| = 0.$$

Further classification:

- stationary problems [fixed time]

$$F(u, \{\partial_x^\alpha u\}_{|\alpha|\leq k}) = 0$$

- evolution problems

$$\partial_t u = F(u, \{\partial_x^\alpha u\}_{|\alpha|\leq k})$$

  → first order in time

  → Also higher order in time

Cauchy problem: add initial data

$$u(t=0, x) = u_0(x)$$

→ time direction is important

→ forward
→ backward $\Big\}$ ⇒ time reversable

3. Solutions to pde's

- solutions in the sense of distrib.

$\mathcal{D} = C_0^\infty$

$\mathcal{D}' =$ space of distributions

- sol's to non linear pde's are functions

Function spaces:

ⓐ L^p spaces, $1 \le p \le \infty$.

BMO - bounded mean oscillation

$L^\infty \subset$ BMO

$$\|u\|_{BMO} = \sup_Q \fint_Q |u - \bar{u}_Q|^p dx$$

$$\bar{u}_Q = \fint_Q u\, dx$$

Hölder spaces C^s, $0 < s < \infty$

Lip = Lipschitz funct.

$C^{k,s}$ $k \in \mathbb{N}$

Sobolev spaces:

$H^{k,p} = \{ u \in L^p, \ \partial^\alpha u \in L^p, \ |\alpha| \le k \}$

$$\| u \|^p_{H^{k,p}} = \sum_{|\alpha| \le k} \| \partial^\alpha u \|^p_{L^p} \quad , \; 1 \le p < \infty$$

$$H^{k,p} = W^{k,p} \quad , \quad W^{1,\infty} = Lip$$

Noninteger Sobolev spaces

$H^{s,p} \to$ defined by interpolation

Special case $p = 2$:

$$H^{s,2} := H^s$$

Sobolev embeddings:

$$H^{s_1,p_1} \subset H^{s_2,p_2}$$

$$\Updownarrow$$

$$\| u \|_{H^{s_2,p_2}} \lesssim C \| u \|_{H^{s_1,p_1}}$$

hidden implicit constant

$\lesssim_A \to$ implicit constant depends on A.

Interpolation inequalities

$$H^{s_0,p_0} \longrightarrow H^{s_h,p_h} \longleftarrow H^{s_1,p_1}$$

$$s_h = (1-h)\, s_0 + h\, s_1$$

$$\frac{1}{p_h} = (1-h) \cdot \frac{1}{p_0} + h \cdot \frac{1}{p_1}$$

$$\|u\|_{H^{s_h,p_h}} \leq \|u\|^{1-h}_{H^{s_0,p_0}} \|u\|^{h}_{H^{s_1,p_1}}$$

$$0 \leq h \leq 1$$

<u>Obs.</u> Embeddings into L^∞ are usually false, but we can use BMO instead.

$\mathbb{R}^d$ $\qquad \underbrace{H^{\frac{d}{2}} \subset L^\infty}_{\text{false}} \qquad H^{\frac{d}{2}} \subset BMO$

Homogeneous Sobolev spaces

$$\dot{H}^{k,p}$$

$$\|u\|^p_{\dot{H}^{k,p}} = \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p}$$

→ Take H^1 for instance:

2 cases: $\quad k < \frac{d}{2} \Rightarrow \dot{H}^k$ is space of distributions
" completion of $\mathcal{D}$

$k \geq \frac{d}{2} \Rightarrow \dot{H}^k$ is a quotient space of distrib modulo constants [or polynomials]

PDE solving:

Stationary: $F(u, \partial^{\leq k} u) = f$

Look for soln. $u \in H^{s,p}$

for well-chosen s, p.

Evolution equation:

$$\begin{cases} u_t + N(\partial_x^{\leq k} u) = 0 \\ u(0) = u_0 \in H^s \end{cases}$$

Def. The above evolution is well-posed in H^s [Hadamard] if for each $u_0 \in H^s$ there exists some $T > 0$ and

→ a solution $u \in C([0,T); H^s)$

→ the solution is unique

→ the solution depends continuously on the data.

$$H^s \ni u_0 \xrightarrow{\text{cont.}} u \in C([0,T); H^s)$$

→ T may depend on data

→ $T = T(u_0)$ is a l.s.c funct.

Fourier analysis

$u \to \mathcal{F}u = \hat{u}$

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int u(x) e^{-ix\xi} dx$$

ξ ↓ Fourier var.

$\mathcal{F} : L^2 \to L^2$

$\mathcal{F} : L^p \to L^{p'}$, $1 \leq p \leq 2$

$\frac{1}{p} + \frac{1}{p'} = 1$.

$$\|u\|_{H^s}^2 = \int |\hat{u}(\xi)|^2 \cdot (1+|\xi|^2)^s d\xi$$

$$\|u\|_{\dot{H}^s}^2 = \int |\hat{u}(\xi)|^2 \cdot |\xi|^{2s} d\xi$$

$\mathcal{F} : \mathcal{S} \to \mathcal{S}$

$\mathcal{S}$ → Schwartz space

$x^\alpha \partial^\beta u \in L^\infty$, $\forall \alpha, \beta$

$\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$

$\mathcal{S}'$ = temperate distributions

$$\mathcal{F}(u \cdot v) = \mathcal{F}(u) * \mathcal{F}(v)$$

Constant coeff pde's

$$\sum_{|\alpha| \leq k} c_\alpha \partial^\alpha u = f$$

$$\widehat{\partial u} = i\xi \cdot \hat{u}$$

$$\partial \to i\xi$$

$$D_j = \frac{1}{i} \cdot \partial_j$$

$P(D)u$ $\qquad D \to \xi$

$\shortparallel$

$$\sum_{|\alpha| \leq k} \tilde{c}_\alpha D^\alpha u = f$$

$$\left(\sum_{|\alpha| \leq k} \tilde{c}_\alpha \xi^\alpha \right) \hat{u} = \hat{f}$$

$$P(\xi) \hat{u} = \hat{f}$$

$\Downarrow$

symbol of our linear partial differential operator.

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{P(\xi)}$$

$\Downarrow$

$$u = f * \left[P(\xi)^{-1} \right]^\vee$$

Works only if $p(\xi) \neq 0$.

$$\check{\left\{p(\xi)^{-1}\right\}} = K$$

$$u = K * f$$

$\downarrow$ fundamental solution for $P(D)$

$$P(D)\, K = \delta_0$$

$\uparrow$ F.T.

$$P(\xi) \cdot \frac{1}{P(\xi)} = 1.$$

Notion of fundamental solution applies even if $P(\xi) = 0$.

Multipliers

$$m : \mathbb{R}^d \to \mathbb{C}$$
$$\xi \to m(\xi)$$

$$\begin{array}{ccc} u & \longrightarrow & m(D)\, u \\ \downarrow \text{F.T.} & & \uparrow \text{IFT} \\ \hat{u}(\xi) & \longrightarrow & m(\xi) \cdot \hat{u}(\xi) \end{array}$$

$$m \in L^\infty \Rightarrow m(D) : L^2 \to L^2$$
$$H^s \to H^s$$
$$\dot{H}^s \to \dot{H}^s$$

$$m(D)\, u = u * \check{m}$$

Symbol type regularity:

$$|\partial^\alpha m(\xi)| \le (1+|\xi|)^{-|\alpha|} \qquad 0 \le \alpha \le N$$

$$\Downarrow$$

$$m(D) : L^p \to L^p \qquad 1 < p < \infty$$

$$m(D) : BMO \to BMO$$

m of order k if:

$$|\partial^\alpha m(\xi)| \le (1+|\xi|)^{k-|\alpha|}$$

<u>Examples</u>

1. H = Hilbert transform:

$$H(\xi) = -i \operatorname{sgn} \xi$$

$$Hu(x) = \mathrm{p.v.}\int \frac{1}{x-y}\, u(y)\, dy$$

2. $|D|^\alpha \to |\xi|^\alpha$

$$\Downarrow$$

Kernel $K^\alpha(x) = c_\alpha \cdot |x|^{-d-\alpha}$

$$\Downarrow$$

$$-d < \alpha < 0.$$

Combine:

$\to$ multiplication: $u \to a(x)\cdot u$

$\to$ multipliers: $u \to m(D)u$

$$\underbrace{a(x)\cdot m(D)}\, u$$

$$b(x,D) = a(x)\, m(D)$$

$$b(x,\xi) = a(x)\, m(\xi)$$

Pseudo-differential operators:

→ start with a symbol $a(x,\xi)$

→ associate an operator

$$a(x,D) : \begin{array}{l} S \longrightarrow S \\ S' \longrightarrow S' \end{array}$$

$$a(x,D)\,u(x) = \int a(\underset{\downarrow\, y}{x},\xi)\, e^{i\xi(x-y)}\, u(y)\,dy\,d\xi$$

$$= \int a(x,\xi)\, e^{ix\xi}\, \hat{u}(\xi)\, d\xi$$

$$a(x,\xi) = \sum a_j(x)\, m_j(\xi)$$

$a(x,D)$ → left quantization

$a(D,y)$ → right quantization

$a\left(D, \frac{x+y}{2}\right)$ → Weyl quantization

Water waves: evolution equations:

Model problem:

$$\begin{cases} \partial_t u = A(D_x)\, u \\ u(0) = u_0 \end{cases}$$

$\Downarrow$ F.T.

$$\begin{cases} \partial_t \hat{u}(\xi) = A(\xi) \cdot \hat{u}(\xi) \\ \hat{u}(0,\xi) = \hat{u}_0(\xi) \end{cases}$$

$$\hat{u}(t,\xi) = e^{tA(\xi)} \cdot \hat{u}_0(\xi)$$

$$u(t) = e^{tA(D)}\, u_0$$

Restrictions on A for well-posedness

$\operatorname{Re} A \leq C \Rightarrow$ well-posedness

→ parabolic problem:

$A(\xi)$ = real, negative

→ solutions have some decay

→ hyperbolic problems:

$A(\xi)$ = purely imaginary

$$\left| e^{tA(\xi)} \right| = 1.$$

$$\| u(t) \|_{L^2} = \| u(0) \|_{L^2}$$

→ rewrite these in the form

$$\begin{cases} i \partial_t u = A(D)\, u \\ u(0) = u_0 \end{cases}$$

→ where $A(\xi)$ = real.