Summer school lecture 2

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Linear "hyperbolic" evolutions

$$\begin{cases} (i\partial_t + A(D))u = 0 & [f] \\ u(0) = u_0 \end{cases}$$

$A(D) \to$ multiplier

$\to a(\xi)$ real valued function

Solve using F.T:

$$\hat{u}(t,\xi) = e^{ita(\xi)}\hat{u}_0(\xi)$$

a real $\Rightarrow \|u(t)\|_{L^2} = \|u_0\|_{L^2}$

Energy is conserved

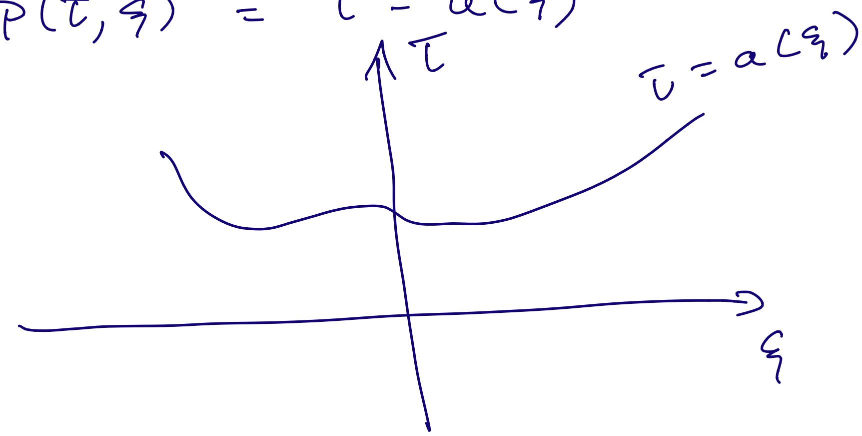
ξ = spatial Fourier variable

τ = time Fourier variable

$$\tau \to D_t = \frac{1}{i}\partial_t$$

Symbol of our operator:

$$P(\tau, \xi) = \tau - a(\xi)$$



Characteristic set of P: $\{\tau = a(\xi)\}$

$a(\xi)$ = dispersion relation for this evolution

Why do we care about char P?

(1) $$u(t,x) = e^{it\tau} e^{ix\xi}$$

solves the eqn. iff $(\tau, \xi) \in \text{char } P$.

(2) Solve

$$[i\partial_t + A(D)]\, u = f$$

$\Downarrow$ F.T.

$$P(\tau, \xi)\, \hat{u}(\tau, \xi) = \hat{f}(\tau, \xi)$$

$$\hat{u}(\tau,\xi) = \hat{f}(\tau,\xi)\cdot\frac{1}{p(\tau,\xi)}$$

$$= \hat{f}(\tau,\xi)\cdot\frac{1}{-\tau + a(\xi)}$$

$$= \hat{f}(\tau,\xi)\cdot\frac{1}{-(\tau \pm i0) + a(\xi)}$$

$$\swarrow$$

$$\lim_{\varepsilon\to 0}\frac{1}{-(\tau \pm i\varepsilon) + a(\xi)}$$

Examples

ⓐ $\quad a = 0$

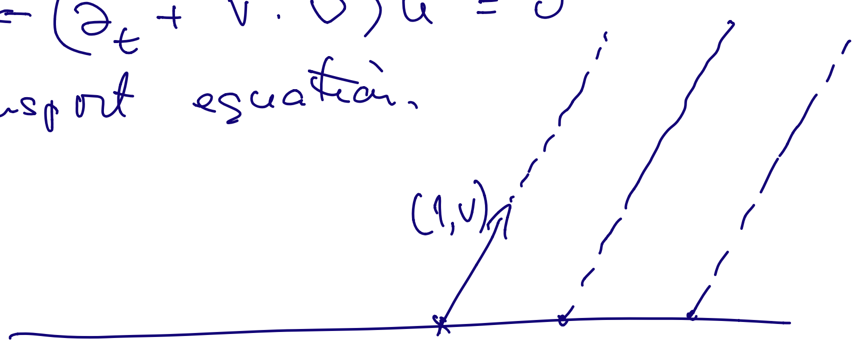
$$i\,\partial_t u = 0$$

ⓑ $\quad a(\xi) = -v\cdot\xi$

$$(i\,\partial_t - v\cdot D_x)u = 0$$

directional derivative $\leftarrow (\partial_t + v\cdot\nabla)u = 0$

Transport equation.



$$u(x,t) = u_0(x - v\cdot t)$$

(c) Schrödinger equation: n-dim.

$$(i\partial_t - \Delta)\, u = 0$$

Symbol of $-\Delta$: $a(\xi) = \xi^2$

(d) Lin KdV equation : 1-dim

$$(\partial_t + \partial_x^3)\, u = 0$$

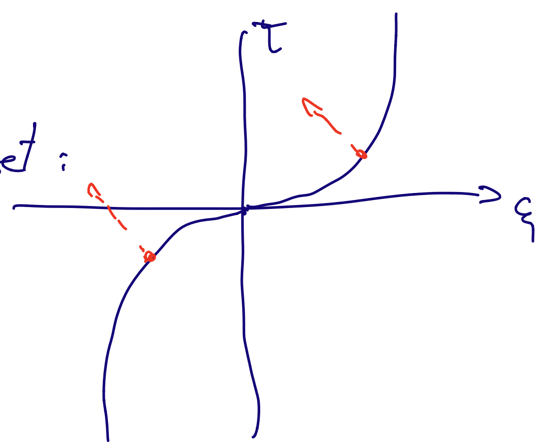
$$a(\xi) = \xi^3$$

(e) lin Benjamin-Ono equation 1-d.

$$(\partial_t + H\partial_x^2)\, u = 0$$

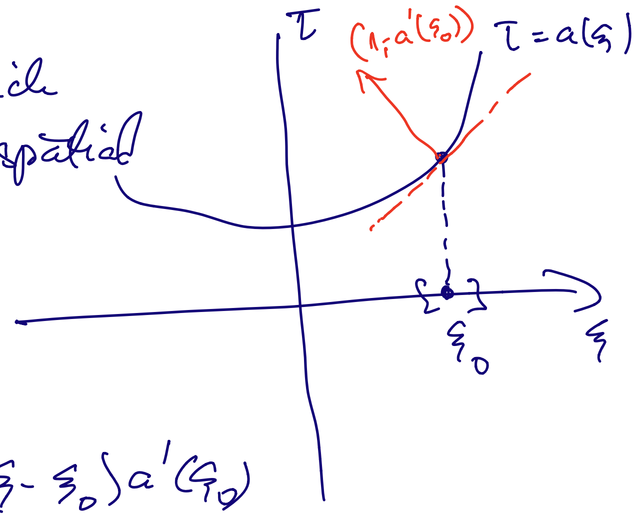
$$a(\xi) = \xi \cdot |\xi|$$

Characteristic set:



Propagation of linear waves

$$(i\partial_t + a(D))u = 0, \quad u(0) = u_0$$



Look at waves which are localized near spatial frequency ξ_0

Near ξ_0:

$$a(\xi) \approx a(\xi_0) + (\xi - \xi_0)a'(\xi_0)$$

Approximate the original evolution

$$(i\partial_t + a(D))u = 0$$

with $(i\partial_t + a(\xi_0) + (D - \xi_0)a'(\xi_0))u = 0$

$$\underbrace{(\partial_t - a'(\xi_0)\cdot\partial_x)}_{\text{transport equation}} u = i\underbrace{(a(\xi_0) - \xi_0 a'(\xi_0))}_{\text{phase shift}}u$$

$$\partial_t u = i\omega u \Rightarrow u(t) = u(0)e^{i\omega t}$$

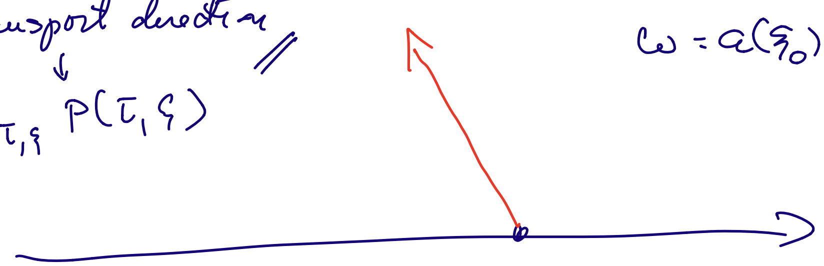
$$\downarrow$$

$$\omega = a(\xi_0) - \xi_0 a'(\xi_0)$$

transport direction $\parallel$ $(1, -a'(\xi_0))$

$\downarrow$

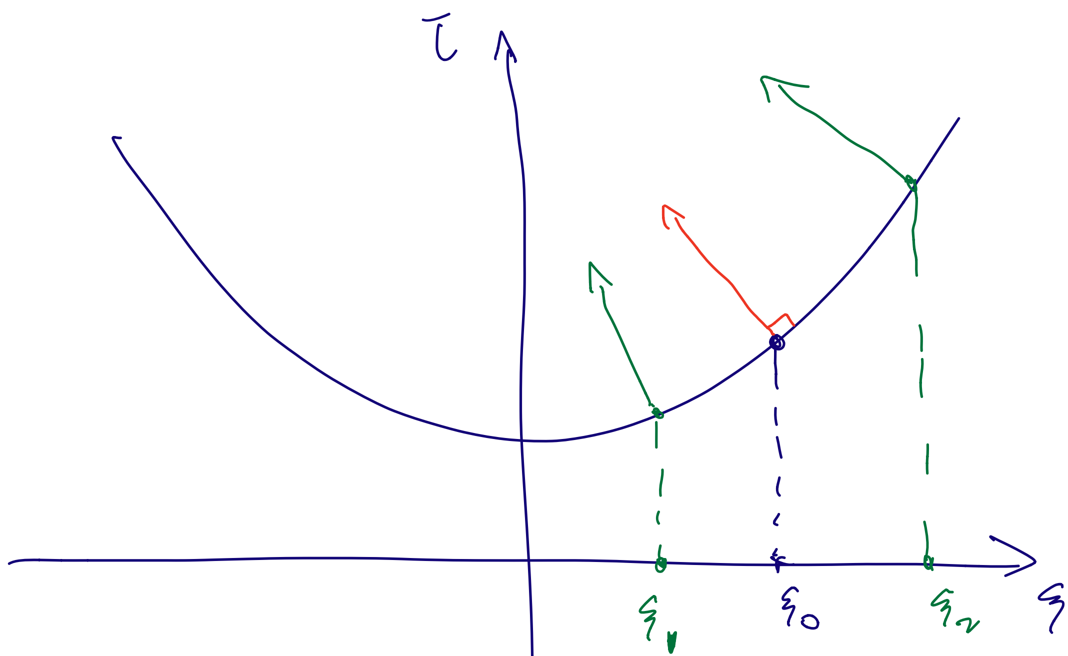
$\nabla_{\tau,\xi} p(\tau,\xi)$



Group velocity:

Waves with frequency $\approx \xi_0$

travel with group velocity $\approx -a'(\xi_0)$



<u>Dispersive equations</u>

"hyperbolic eqn's" where the group velocity depends nontrivially on the frequency.

$$v = U(\xi) = -a'(\xi), \text{ has to vary with } \xi$$

Full dispersion $\Longleftrightarrow$ $a''(\xi) \neq 0$

Systems of equations:

$$\begin{cases} (i\partial_t + A(D))u = 0 \\ u(0) = u_0 \end{cases}$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \qquad A \in M^{m\times m}, \; A = A(\xi)$$

Symmetric systems $\rightarrow$ A = symmetric

Hyperbolic system: A real symmetric matrix

$$(i\partial_t + \lambda_j(D))u_j = 0$$

eigenvalues

System for which A has real eigenvalues.

Strictly hyperbolic systems:

$\lambda_j \rightarrow$ real and distinct.

$\rightarrow$ If eigenvalues are distinct then the system can be smoothly diagonalized in the Fourier space.

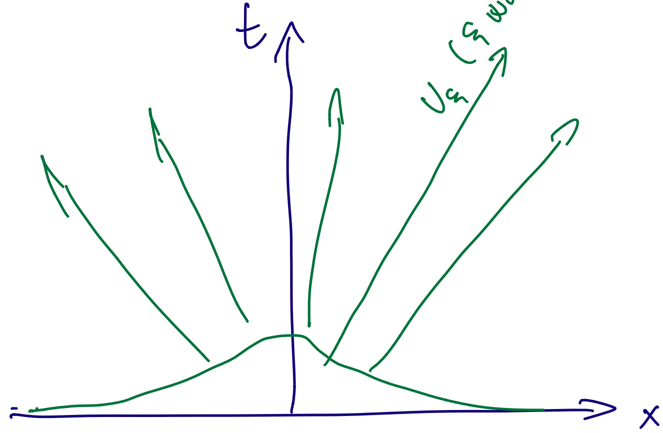
Return to scalar equations

Q: Long time behavior of linear waves.

$$(i\partial_t + A(\partial))u = 0$$

$$u(0) = u_0 \in S$$

$$\xi \to v_\xi$$



Measure the dispersive decay:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi) \cdot e^{ita(\xi)}$$

$$u(t,x) = \int e^{ita(\xi)} e^{ix\xi} \underbrace{\hat{u}_0(\xi)}_{S} d\xi$$

$$v = \frac{x}{t}$$

$$= \int e^{i(ta(\xi) + x\xi)} \hat{u}_0(\xi)\, d\xi$$

$$= \int e^{it(a(\xi) + v\xi)} \hat{u}_0(\xi)\, d\xi$$

Method of stationary phase

$$I = \int e^{i\lambda\phi(\xi)} \cdot a(\xi)\, d\xi$$

a) $\phi' \neq 0$

$$I = \int \frac{1}{\phi'} a(\xi) \underbrace{e^{i\lambda\phi}\cdot\phi'}_{\frac{1}{\lambda}\partial_\xi e^{i\lambda\phi}} d\xi$$

Repeated integrations by parts:

$$\Longrightarrow \boxed{I = O(\lambda^{-N}) \quad \forall N.}$$

(b) $\phi'(\xi_0) = 0$, $\phi''(\xi_0) \neq 0$

Main contribution comes from $\xi \simeq \xi_0$

$$I \simeq \int e^{i\lambda(\phi(\xi_0) + \frac{1}{2}(\xi-\xi_0)^2\phi''(\xi_0))} a(\xi_0) d\xi_0$$

$$I = e^{i\lambda\phi(\xi_0)} \cdot a(\xi_0) \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{|\phi''(\xi_0)|^{1/2}} \cdot e^{\frac{i\pi}{4}\operatorname{sgn}\phi''}$$

$$u(t,x) = \int e^{it\overbrace{(a(\xi) + v\xi)}^{\phi}} \hat{u}_0(\xi)\, d\xi$$

$$a'(\xi_v) = -v \longrightarrow \text{critical points}$$

$$u(t, \underset{x}{\uparrow}{vt}) \approx \frac{1}{\sqrt{t}} \hat{u}_0(\xi_v) \cdot \frac{1}{|a''(\xi_v)|^{1/2}} \cdot e^{it(a(\xi_v) + v\xi_v)}$$

Comments about the phase:

(1) Different ways of writing the phase:

$$\phi(\xi_v) = \left[a(\xi_v) + v\,\xi_v\right]\cdot t = \psi(t,x)$$

$$= \gamma(v)\cdot t$$

Try to relate: $a(\xi) \longleftrightarrow \gamma(-v)$

Legendre transform $\longrightarrow$ Evans p. 120

(2) Connection to eikonal equation
(often seen in the context of the wave eqn)

$$\left(i\partial_t + a(D)\right) e^{i\psi(x,t)} \approx 0$$

$$\downarrow \qquad [\tau] \qquad [\xi]$$

the exponential has freq. (ψ_t, ψ_x)

$$\boxed{-\psi_t + a(\psi_x) = 0}$$ eikonal equation

It is a good exercise to verify that the eikonal equation has solutions of the form

$$\psi(t,x) = t\,\gamma(-v), \qquad v = \frac{x}{t}$$

where γ is as in (1) above.