# Summer School Lecture 3

Daniel Tataru

Linear dispersive flows

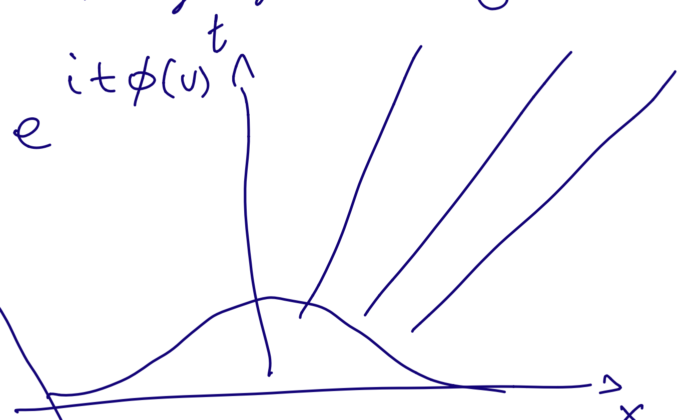
$$\begin{cases} i u_t + A(D) u = 0 \\ u(0) = u_0 \end{cases}$$

$a(\xi) \to$ dispersion relation

$\xi \longrightarrow -a'(\xi)$ group velocity

Asymptotic as $t \to \infty$:

$$u(t) \simeq \frac{1}{\sqrt{t}} \cdot c(V) \cdot e^{it\phi(v)}$$




connected to the fact that $a''(\xi) \neq 0$.

depends on data, size of $a''(\xi)$.

<u>Remark</u> In $n$-dim. we have $t^{-\frac{n}{2}}$ decay of linear waves provided that the Hessian $D^2 a(\xi)$ is non degenerate.

Model problem: Schrodinger equation

$$(i\partial_t + \Delta) u = 0$$

$$u(0) = u_0$$

$a(\xi) = \xi^2$ $\qquad D^2 a = 2 I_n$

$\longrightarrow$ uniform $t^{-\frac{n}{2}}$ decay for waves with localized data

$\Downarrow$

Dispersive estimates (dispersive decay)

$$\| u(t) \|_{L^\infty} \le \frac{c}{t^{\frac{n}{2}}} \| u(0) \|_{L^1}$$

<u>Remark</u>. For other equations, size of $D^2 a$ may depend on size of $\xi$.

So to write dispersive estimates it is useful to localize in frequency.

What if data is in $L^2$?

$$\| u(0) \|_{L^2} = \| u(t) \|_{L^2}$$

Replace uniform decay by averaged decay.

<u>Strichartz estimates</u>

Strichartz for Schrodinger:

1-d Energy:

homogeneous est.

$$\begin{cases} \|u\|_{L^\infty_t L^2_x} \leq \|u_0\|_{L^2} \\ \text{Endpoint Strichartz:} \\ \|u\|_{L^4_t L^\infty_x} \leq \|u_0\|_{L^2} \\ \|u\|_{L^6_t L^6_x} \leq \|u_0\|_{L^2} \end{cases}$$

can interp.

$$S = L^\infty L^2 \cap L^4 L^\infty$$

$$\|u\|_S \leq \|u_0\|_{L^2}$$

Inhomogeneous problem:

$$\begin{cases} iu_t + \Delta u = f \in S' = L^1 L^2 + L^{4/3} L^1 \\ u(0) = 0 \end{cases}$$

$$\|u\|_S \leq \|f\|_{S'} \quad \text{[Inhom. Strichartz]}$$

Dispersive est. $\Longrightarrow$ Strichartz [Tao]

n-d 2-d

$$S = \begin{cases} L^\infty_t L^2_x \\ L^4_t L^4_x \\ \cancel{L^2_t L^\infty_x} \end{cases}$$

$\to$ forbidden endpoint

# Semilinear dispersive equations

Model problem:

$$\begin{cases} (i\partial_t - \Delta)u = \pm u|u|^2 \\ u(0) = u_0 \end{cases} \quad \text{in 2-d.}$$

$\pm$ : focusing / defocusing character

Conservation laws:

Mass: $M = \frac{1}{2}\int |u|^2 dx$

Energy $E = \int \frac{1}{2}|\nabla u|^2 \mp \frac{1}{4}|u|^4 dx$

$-$ defocusing

$+$ focusing

Q: Local well-posedness:

$$\begin{cases} (i\partial_t - \Delta)u = f \\ u(0) = u_0 \end{cases}$$

$$u(t) = e^{-it\Delta}u_0 + \int_0^t e^{-i(t-s)\Delta} f(s)\,ds$$

Duhamel formula.

For nonlinear problem:

$$u(t) = e^{-it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta} u \cdot |u|^2(s)\,ds$$

Standard strategy: Use a fixed point argument (Contraction principle) to solve this equation.

$$u \to Lu$$

$$Lu(t) = e^{-it\Delta} u_0 + \int_0^t e^{i(t-s)\Delta} (u \cdot |u|^2)(s) ds$$

Solving the equation $\Leftrightarrow$ Fixed point for $L$:

$$u = Lu$$

By contraction principle.

We need a Banach space $X$, some ball $B \subset X$, such that

(1) $L : B \to B$

(2) $L$ is a contraction

$$\|Lu - Lv\|_B \le \gamma \|u-v\|_B$$

$\gamma < 1$.

$\boxed{\text{Take 1:}}$ Use energy estimates:

For the inhom. problem:

$$\|u\|_{H^s} \le \|u_0\|_{H^s} + \int_0^t \|f(s)\|_{H^s} ds$$

$$X = L^\infty(0,T; H^s)$$

$$\|Lu - Lv\|_{L^\infty(0,T; H^s)} \le \int_0^T \|u|u|^2 - v|v|^2\|_{H^s} ds$$

Sobolev embeddings,

$$H^s \subset L^\infty \quad \text{if} \quad \underline{\underline{s > 1}} \quad (2\text{-d})$$

Fixed time bound:

$$\| u|u|^2 - v|v|^2 \|_{H^s} \leq \|u-v\|_{H^s} \big( \underbrace{\|u\|_{H^s} + \|v\|_{H^s}}_{} \big)^2$$

$$\| Lu - Lv \|_{L^\infty H^s} \leq T \, \|u-v\|_{H^s} \cdots - - -$$

Choosing $T$ small makes $\delta$ small.

When we can solve the equ. by a fixed point argument $\Rightarrow$ Lipschitz dependence of solutions on initial data:

$$\|u-v\|_{L^\infty H^s} \lesssim \|u_0 - v_0\|_{H^s}$$

Defining feature of semilinear problems!

Scaling symmetry:

$$u(x,t) \Rightarrow \lambda \, u(\lambda x, \lambda^2 t)$$

2-d: Critical Sobolev space

$$X = L^2$$

$\boxed{\text{Take 2:}}$ Use Strichartz estimates:

<u>Theorem.</u> $NLS_3$ in 2-d is locally well-posed in $L^2$

$$\underbrace{u(t)}_{L^4} = \underbrace{\underbrace{e^{-it\Delta} u_0}_{\downarrow \, L^2}}_{S \,\cap\, L^4} + \int_0^t e^{-i(t-s)\Delta} \underbrace{u \cdot |u|^2}_{L^{4/3} \,\cap\, S'} ds$$

(dual Strichartz)

Fixed point argument in $L^4$.

Small issue: no time in Strichartz est.

Easy fix: Take $\|u_0\|_{L^2} \ll 1$.

$\Rightarrow$ $\boxed{\text{Global solution for small data.}}$

Pull back $u(t)$ to time $0$:

$$e^{it\Delta} u(t) = u_0 + \underbrace{\int_0^t e^{is\Delta} \underbrace{u|u|^2}_{L^3} ds}_{\text{convergent in } L^2}$$

$\Downarrow$

has a limit as $t \to \infty$.

$\lim_{t \to \infty} e^{it\Delta} u(t) = u_+$ exists in $L^2$.

$$\lim_{t \to \infty} u(t) - \underbrace{e^{-it\Delta} u_+}_{\text{linear flow of } u_+} = 0 \quad \text{in } L^2$$

$$u_0 \to u_+$$

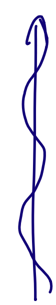
Scattering map.

**Def.** $u$ is scattering at $\infty$ if it gets closer to a solution for the linear equation.

Focusing problems admit solitons:

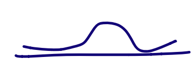
- stationary solutions

$$\Delta u = u \cdot |u|^2$$

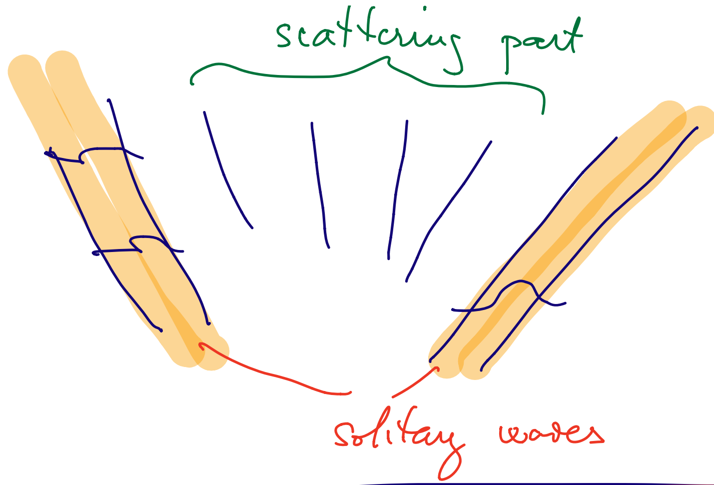


- soliton: (with added phase shift)

$$\Delta u = u|u|^2 + \omega u$$



- solitary waves (with added velocity)




Soliton resolution conjecture:
A global solution "resolves" into a collection of solitons and a dispersive part.

→ too vague as stated
→ false in many cases
→ proved in very few, completely integrable cases
→ take as a question rather than as conjecture.
→ can be applied to many water wave models.