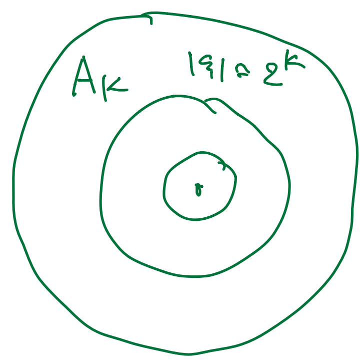
Summer School Lecture 4

Daniel Tataru

① Littlewood Paley theory

Fourier space $\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}} \underbrace{\{ |\xi| \approx 2^k \}}_{A_k}$

$$1 = \sum \phi_k(\xi)$$



supp $\phi_k \subset A_k$

ϕ_k smooth on 2^k scale

$$|\partial^\alpha \phi_k| \leq 2^{-k|\alpha|}$$

Possibly start with ϕ_0, and then set

$$\phi_k(\xi) = \phi_0\left(\frac{\xi}{2^k}\right)$$

Partition of unity for multipliers

$$Id = \sum \phi_k(D) \qquad (LP)$$

Notation: $\phi_k(D) = P_k$

Littlewood - Paley decomposition of u

$$u = \sum u_k := \sum P_k u$$

In L^2:

$$\|u\|_{L^2}^2 \approx \sum \|u_k\|_{L^2}^2$$

u_k are almost orthogonal

$u \in \dot{W}^{s,p}$, look at $P_k u$.

$$2^{sk} \cdot \|P_k u\|_{L^p} \leq \|u\|_{\dot{W}^{s,p}}$$

$$u \longrightarrow \{u_k\}$$

measure in L^p

add 2^{ks} for s derivatives

$$\| 2^{sk} \|P_k u\|_{L^p} \|_{\ell^q_k} = B^s_{p,q}$$

<u>Besov spaces</u>

$$\| \| 2^{sk} P_k u \|_{\ell^q_k} \|_{L^p_x} = F^s_{p,q}$$

<u>Triebel - Lizorkin spaces</u>

Thm

$$\dot{W}^{s,p} = F^{s}_{p,2} \qquad 1 < p < \infty.$$

$$BMO = F^{0}_{\infty,2}$$

Notation $\quad BMO^{s} = F^{s}_{\infty,2}$

$$C^{\gamma} = B^{\gamma}_{\infty,\infty}.$$

$q = 2$

$\Downarrow$

$$\left[\sum \left(2^{sk} P_k u\right)^2\right]^{1/2} = S_k(u)$$

$\downarrow$

square function of u

(2) Sobolev spaces $\Rightarrow$ Sobolev embeddings

$$W^{s_1,p_1} \subset W^{s_2,p_2}$$

$$\|u\|_{W^{s_2,p_2}} \le \|u\|_{W^{s_1,p_1}}$$

For L-P projections:

$$\underbrace{2^{ks_2} \|P_k u\|_{L^{p_2}} \le 2^{ks_1} \|P_k u\|_{L^{p_1}}}$$

Bernstein inequalities,

Proved by Young's convolution inequality.

(2) Bilinear analysis

$$\|u \cdot v\|_{L^r} \le \|u\|_{L^p} \cdot \|v\|_{L^q}$$

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

$$u \cdot v = \sum_{k,j \in \mathbb{Z}} \underset{\substack{\downarrow \\ 2^k \text{ freq}}}{u_k} \cdot \underset{\substack{\downarrow \\ 2^j \text{ freq.}}}{v_j}$$

Littlewood-Paley trichotomy

$$= \sum_{k<j} \underbrace{u_k \quad v_j}_{\text{low} \times \text{high}} \to \text{at freq } 2^j$$

$$+ \sum_{j<k} \underbrace{u_k \quad v_j}_{\text{high} \times \text{low}} \quad \text{at freq } 2^k$$

$$+ \sum_{j=k} \underbrace{u_k \quad v_k}_{\text{high} \quad \text{high}} \quad \text{at freq.} \le 2^k$$

Ex: Multiply $H^{s_1} \cdot H^{s_2}$

→ LP trichotomy

→ Bernstein inequalities

$$uv = \sum_k \underbrace{u_{<k} v_k + u_k v_{<k}}_{\text{paraproducts}} + u_k v_k$$

$$uv = \sum_k T_u v + T_v u + \Pi(u,v)$$

Coifman – Meyer estimates:

$$\|T_u v\|_{L^r} \leq \|u\|_{L^p} \|v\|_{L^q}$$

$$\|T_u v\|_{L^p} \leq \|u\|_{L^p} \|v\|_{BMO}$$

$$\|\Pi(u,v)\|_{L^p} \leq \|u\|_{L^p} \|v\|_{BMO}$$

$$(u, v) \to T_u v$$

- pseudodif. operators op
    - based on left quantization
    - Weyl quantization also useful
    - between operator translation invariant

Translation invariant bilinear forms

$$\widehat{u \cdot v}(\zeta) = \int_{\xi+\eta=\zeta} \hat{u}(\xi)\, \hat{v}(\eta)\, d\xi$$

$$u \cdot v(x) = \int e^{ix(\xi+\eta)} \cdot \underset{\uparrow}{\hat{u}(\xi)}\, \underset{\uparrow}{\hat{v}(\eta)}\, d\xi d\eta$$

$m(D)u \cdot p(D)v$ $\qquad m(\xi \quad , \quad p(\eta)$

$$M(u,v) = \int e^{ix(\xi+\eta)} \underbrace{m(\xi,\eta)}_{\text{symbol}} \hat{u}(\xi)\, \hat{v}(\eta)\, d\xi d\eta$$

Representation for all translation invariant bilinear forms!

<u>Commutators</u>

$$[\underset{\substack{\downarrow \\ \text{order } 0}}{m(D)}, f]\, g = m(D)(f \cdot g) - f\, m(D) g$$

$$[\partial, f]\, g = \partial f \cdot g$$

$\Downarrow$

better if

freq of f < freq.

$$[m(D), f]\, g = [m(D), T_f]\, g + [m(D), R_f]\, g$$

commutator matters. commutator does not matter

$$[m(D), T_f]\, g \approx \underbrace{m'(D)\, T_{f'}\, g}$$

$$L_{lh}(f', g) \leftarrow B(f', g)$$

$L_{lhe} \longrightarrow$ trilinear form.

4. Frequency envelopes.

$u \in L^2$, $\quad u = \sum u_k$

$$\|u\|_{L^2}^2 \approx \sum \|u_k\|_{L^2}^2$$

$$\cancel{\|} \; \le$$

$$c_k$$

Definition: c_k is an L^2 frequency envelope for a function u if:

① $\|u_k\|_{L^2} \le c_k$

② c_k slowly varying:

2 c_k slowly varying.

$$\left|\frac{c_k}{c_j}\right| \le 2^{\delta|k-j|}$$

3 $\|c_k\|_{\ell^2} \approx \|u\|_{L^2}$

$$\begin{array}{ccccc} u\in X & , & v\in Y & \Longrightarrow & uv\in Z\\ \downarrow & & \downarrow & & \downarrow\\ c_k & & d_j & & ?\\ & & & & \Downarrow \end{array}$$

see homework
problem for
example

5. Nonlinear bounds:

$$\begin{array}{ccc} u & \longrightarrow & F(u) \\ \downarrow & & \downarrow \\ H^s & & ? \end{array}$$

<u>Moser inequalities</u>

$u \in H^s$, $s > \frac{d}{2} \Rightarrow F(u) \in H^s$

$u \in H^s$, $s \geq 0 \Rightarrow F(u) \in H^s$
$u \in L^\infty$

What is the underlying inequality:

$$\|F(u)\|_{H^s} \leq C(\|u\|_{L^\infty}) \|u\|_{H^s}$$

More accurate question: What is the leading part of $F(u)$?

$$\begin{array}{ccc} u & \cdot & v \\ \downarrow & & \downarrow \\ H^s & & H^s \end{array} \quad \begin{array}{l} \left.\begin{array}{l} \text{low - high} \\ \text{high - low} \end{array}\right\} \text{paradiff.} \\ \text{high - high} \to \text{better bounds} \end{array}$$

Thm [Bony]

$$F(u) = T_{F'(u)} u + R$$

$\Downarrow$ better.

$u \to u_{<h}$

$\Downarrow$ freq. below 2^h (regularization of u)

$$\lim_{h\to\infty} u_{<h} = u$$

$$F(u) = \lim_{h\to\infty} F(u_{<h})$$

$$= F(u_{<0}) + \int_0^\infty \frac{d}{dh} F(u_{<h})\, dh$$

$$= F(u_{<0}) + \int_0^\infty F'(u_{<h}) \cdot u_h \, dh$$

$$u_h = \frac{d}{dh} u_{<h}$$

expand further

$$F^{(k)}(u_{<h}) \cdot u_h \; u_{h_1} \cdots u_{h_k}$$

Frequency order $<$ $<$ $<$