Summer School Lecture 5

Daniel Tataru

Well-posedness for quasilinear evolution equations.

General model:

$$(NL)\quad \begin{cases} u_t = N(u) \\ u(0) = u_0 \in H^s \end{cases}$$

$$N(u) = N\left(\{\partial^\alpha u\}_{|\alpha|\le k}\right)$$

k = order of our system

<u>Def</u>. (NL) is well-posed in H^s in the Hadamard sense if:

1. A solution $u \in C(0,T; H^s)$ exists
2. The solution is unique.
3. The solution depends continuously on the initial data u_0.

$$H^s \ni u_0 \longrightarrow u \in C(0,T; H^s)$$

Best T = lifespan of solutions.

$$T = T(u_0)$$

We expect that

$T(u_0)$ = lower semicontinuous function of u_0

$$\underbrace{u_0^n \to u_0}_{H^s} \Rightarrow \liminf_{n\to\infty} T(u_0^n) \geq T(u_0)$$

- first look for WP in high Sobolev spaces
- ask for low regularity solutions, look for WP for low s

s_c - critical s
$\Downarrow$
hard threshold

often together

- get good life span bounds
$$T(u_0) \geq C(\|u_0\|_{H^s})$$
- Blow up criteria
  - simplest: the solution blows up if $\|u(t)\|_{H^s} \to \infty$.

  (but not the best by far)

Model problem

- symmetric hyperbolic system

$$(SH) \qquad u_t = A^j(u)\, \partial_j u$$

$$u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m \ , \quad A^j \in M^{m \times m}$$

$$u(0) = u_0 \in H^s \qquad \text{real, symmetric.}$$

Scaling symmetry:

$$u(x,t) \to u(\lambda x, \lambda t)$$

Scale invariant Sobolev space

$$\dot{H}^{\frac{d}{2}} \ , \quad s_c = \frac{d}{2}$$

__Theorem__ (SH) is locally well-posed in H^s for $s > \frac{d}{2} + 1$.

- Hadamard style well-posedness.

Return to general problem.

$$u\,v = \underset{l\ h}{T_u v} + \underset{h\ l}{T_v u} + \underset{h\ h}{\Pi(u,v)}$$

better behaved at high regularity

$$u_t = N(u)$$

$\Updownarrow$

$DN \to$ linearization of N

$v \to DN(u)\,v$

[directional derivative]

$$u_t = \underbrace{T_{DN(u)}}_{l\ h} u + \underbrace{F(u)}_{h\ h}$$

$\Downarrow$

perturbative

Paradifferential equ.

$$w_t = T_{DN(u)}\, w \quad , \quad w(0) = w_0$$

Frequency localized:

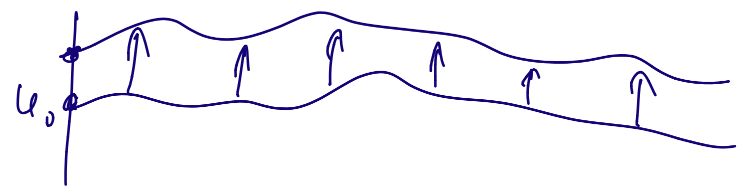
$$w_{kt} = T_{DN(u_{<k})}\, w_k + F_k(w)$$

Linear equation with variable coeff.

[high freq. solution, moving on a low freq. background]

Linearized equation

$$v_t = \underbrace{DN(u)}_{\text{Diff. operator of order } k} v \quad , \quad v(0) = v_0$$



Paradifferential form

$$v_t = T_{DN(u)} v + F^{lin}(u) \cdot v$$

Model problem :

$$u_t = A(u)\, \partial u$$

$$N(u) = A(u)\, \partial u$$

$$DN(u) v = A(u)\, \partial v + DA(u) \cdot v \cdot \partial u$$

Paradiff. eqn :

$$w_t = \underbrace{T_{A(u)}}_{\text{first order}} \partial w + \underbrace{T_{DA(u) \cdot \partial u}}_{\text{order } 0} w$$

Linearized equation :

$$v_t = A(u)\, \partial v + DA(u) v \cdot \partial u$$

Steps in proving local well-posedness

1. Energy estimates
2. Existence of solutions
3. Uniqueness $\Leftarrow$ difference estimates
$\Updownarrow$
linearized eqn. bounds
4. Continuous dependence
   - frequency envelope bounds.
   - cont. dependence.

Today: Energy estimates

- track the H^s norm of u
- also want to track H^σ norm of u, for $\sigma \geq 0$.

$\sigma > s \implies$ higher regularity

Naively:

$$\frac{d}{dt} \|u\|_{H^\sigma}^2 \leq C \|u\|_{H^\sigma}^2$$

$C \to C(\|u\|_{H^s})$

Remarks:

(1) In general

$$\|u\|^2_{H^\sigma} \longrightarrow E^\sigma(u)$$

(2) Better if we can control C using some pointwise measurements of the solution.

$A \rightarrow$ "scale invariant size"

$B \rightarrow$ pointwise norm, replacement for C.

Overall, we want energy functionals E^σ with properties:

(1)
$$E^\sigma(u) \approx_A \|u\|^2_{H^\sigma}$$

(2)
$$\frac{d}{dt} E^\sigma(u) \leq_A B \|u\|^2_{H^\sigma}$$

Linearized equation bound in L^2

$$\frac{d}{dt} E^{0,\text{lin}}(v) \lesssim_A B \quad \|v\|^2_{L^2}$$

Model problem

$$u_t = A(u) \cdot \partial u$$

$$A = \|u\|_{L^\infty}$$

$$B = \|\nabla u\|_{L^\infty}$$

Linearized eqn:

$$v_t = A(u) \cdot \partial v + DA(u)\, v \cdot \partial u$$

$$\frac{d}{dt} \|v\|_{L^2}^2 = \int v \cdot A(u) \cdot \partial v + v^2 DA(u) \partial u \, dx$$

$$= \int -\frac{1}{2} v^2 \cdot \partial A(u) + v^2 \cdot DA(u) \cdot \partial u \, dx$$

$$\lesssim \|v\|_{L^2}^2 \underbrace{\|\partial u\|_{L^\infty}}_{B} \cdot C(A)$$

Full equation:

$\underbrace{\partial^k u}_{w}$ solves the equation

$$\partial_t w = A(u) \cdot \partial w + \underbrace{G(u)}$$

Multilinear

$K+1$ derivatives

No term with

$K+1$ derivatives

$$a(u) \cdot \partial u \cdot \partial^K u$$

$$a(u) \cdot \partial^2 u \cdot \partial^{K-1} u$$

$$\frac{d}{dt} \|w\|_{L^2}^2 \le B \|w\|_{L^2}^2 + \|w\|_{L^2} \cdot \|G\|_{L^2}$$

$$\|w\|_{L^2}^2 = \|u\|_{H^K}^2$$

Claim: $\|G(u)\|_{L^2} \lesssim_A B \|u\|_{H^K}$

$$\|\partial^2 u \, \partial^{K-1} u\|_{L^2} \le \|\partial^2 u\|_{L^p} \cdot \|\partial^{K-1} u\|_{L^q}$$

Interpolate between

$\|\partial^K u\|_{L^2}$

and

$\|\partial u\|_{L^\infty} = B$