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Lecture 1

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Plan: Eulerian set-up & Sobolev existence
Lagrangian set-up
Regularity in Lagrangian vs. Eulerian setting
Local existence for free-surface Euler

Euler equations

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = 0 \\ \nabla\cdot u = 0 \end{cases}$$

$u\cdot\nabla u = u_j \partial_j u$

$(j = 1, \ldots, d)$

Initial condition

$$u|_{t=0} = u_0$$

EE incompr. inv. fluid

1) $\mathbb{R}^N$

2) $\mathbb{T}^N$

3) $\Omega \subseteq \mathbb{R}^N$ bdd

with $u\cdot n = 0$ on $\partial\Omega$

Also interesting: $\Omega = \mathbb{R}^d_+$ or channel

The first eq. comes from the Newton's limit

mat. derivative of u = forces

Div. free free cond. comes from

$$\rho_t + \mathrm{div}\,(u\cdot\rho) = 0$$

by setting $\rho \equiv 1$.

Great book: Majda–Bertozzi.

Existence & uniqueness: $W^{m,p}$, $m > \frac{d}{p}+1$

Loc. Well-posedness: $u \in C^{1,\alpha}$ $\quad \alpha \in (0,1)$

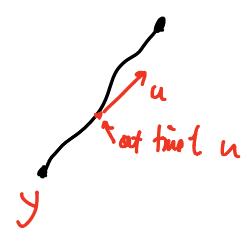
Particle Trajectory

Let u be a v.f. Consider particle traj.

$$\eta(\cdot,t) : y \longmapsto \eta(y,t) \qquad (*)$$

by for y fixed

$$\begin{cases} \dfrac{d\eta}{dt}(y,t) = u(\eta(y,t),t) \\ \eta(y,0) = y \end{cases}$$



y Lagrangian label. Denote

$$J = \det(\nabla\eta)$$

Proposition u is a smooth v.f. Then

$$\partial_t J = \mathrm{div}\, u(\eta(y,t),t)\, J$$

div $u = 0$: $\quad J \equiv 1$

Proof : $\quad \partial_t (\nabla_y \eta) = \nabla u \, \nabla \eta$

Proposition (the Transport Theorem)

$\Omega \subseteq \mathbb{R}^d$ smooth domain, u smooth v.f.,

η is a particle traj map. For any smooth f, we have

$$\frac{d}{dt} \int_{\eta(\Omega, t)} f \, dx = \int_{\eta(\Omega, t)} (\partial_t f + \operatorname{div}(f u)) \, dx$$

$f = \rho$ density : $\quad \partial_t \rho + \operatorname{div}(u \rho) = 0$

$\rho \equiv 1$: $\quad \operatorname{div} u = 0$

Proof

$$\frac{d}{dt} \int_\Omega f(\eta, t) \, J \, dy$$

$$= \int_\Omega \Big(\partial_t f + \nabla f \, \partial_t \eta J + f \underbrace{\partial_t J}_{\operatorname{div} u \, J}\Big).$$

Lagrangian form : $\quad v(y, t) = u(\eta(y, t), t)$

Local Existence

(Sketch) The Euler eq. keep L^2 norm of u constant

$$\partial_t u + u \cdot \nabla u + \nabla p = 0 \qquad \Big| \cdot u$$

$$\int u_j \underbrace{\partial_j u_k \, u_k}_{\frac{1}{2}\partial_j (u_k u_k)} = 0 \qquad (\text{in } \Omega : u \cdot \nu = 0)$$

$$\int \partial_k p \, u_k = 0 \qquad (\text{by } \nabla \cdot u = 0)$$

Well-posedness in $C^{1,\alpha}$ $0<\alpha<1$: Lichtenstein '25, Gunther '27

H^m, $m > \frac{d}{2}+1$: Ebin-Marsden '70,

Bourguignon-Brezis '84

Temam '86

<u>A-priori estimates</u> showing existence for $m=3$, $d=3$

(assume sol'n exists, smooth, w. suff. decay; find bounds based on initial data)

For $|\alpha| \leq 3$:

$$\partial_t \partial^\alpha u_k + \partial^\alpha (u_j \partial u_k) + \partial_k \partial^\alpha p = 0 \qquad \Big| \cdot \partial^\alpha u_k, \int, \sum$$

$$\frac{1}{2}\frac{d}{dt} \sum_{|\alpha| \leq 3} \int \partial^\alpha u_k \, \partial^\alpha u_k + \sum_{\substack{|\alpha| \leq 3 \\ 0 < \beta \leq \alpha}} \int \partial^\beta u_j \, \partial_j \partial^{\alpha-\beta} u_k \, \partial^\alpha u_k = 0$$

Since $\int u_j \, \partial_j \partial^\alpha u_k \, \partial^\alpha u_k = 0$ by $\nabla \cdot u = 0$

By Hölder and Sobolev

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^3}^2 \leq C \|u\|_{H^3}^3$$

A-priori estimates for local existence.

$$\| \partial^\alpha (uv) - u \, \partial^\alpha v \|_{L^2} \lesssim \|u\|_{H^3} \|v\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|v\|_{H^3}$$

Kato - Ponce ineq. $(m > \frac{d}{p} + 1)$

α integer : Leibnitz rule

$T = (I - \Delta)^m$, $m > \frac{d}{2} + 1$

Kato - Ponce ineq. also hold in L^p

(Bourgain - Li)

To construct sol'ns, we can use Galerkin ($\mathbb{R}^n$)

$$u_t + P_m (u \cdot \nabla u) + P_m \nabla p = 0$$

where P_m is proj. on spectral space.

More common:

$$u_t + (u_j)_\varepsilon \cdot \nabla u + \nabla p = 0$$

where $v_\varepsilon = v * \rho_\varepsilon$, $\rho_\varepsilon = \frac{1}{\varepsilon^n} \rho \left(\frac{\cdot}{\varepsilon} \right)$.

2D Yudovich data : Global ex. & uniq.

$\omega_0 = \operatorname{curl} u_0 \in L^1 \cap L^\infty$

M. Vishik : fails in $L^1 \cap L^p$, $p < \infty$

3D global existence : unknown

BKM: If $T_{max} < \infty$: $\int^{T_{max}} \| \omega \|_{L^\infty} = \infty$

Elgindi: Global existence may fail in $C^{1,\alpha}$

NSE: C^2 data : global existence of Leray weak sol'ns

EE: this is unknown

$\exists$ measure-valued sol'ns

Vorticity Transport Formula

Introduce

$$\omega = \operatorname{curl} u$$

(motivation : $\operatorname{curl} \nabla p = 0$)

In 2D:

$$\omega = \partial_1 u_2 - \partial_2 u_1$$

In 3D:

$$\omega = \begin{bmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{bmatrix}$$

$$\left(\Omega = \tfrac{1}{2}\left(\nabla u - (\nabla u)^T\right)\right)$$

To get an equation for ω in 3D, use

$$u \cdot \nabla u = \frac{1}{2} \nabla(|u|^2) - u \times \operatorname{curl} u$$

We get in 2D:

$$\omega_t + u \cdot \nabla \omega = 0$$

in 3D

$$\omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u.$$

To compute ω:

$$\operatorname{div} u = 0$$
$$\operatorname{curl} u = \omega$$
$$(u \cdot n = 0)$$

To solve for u: curl of 2nd eq.

$$-\Delta u = \operatorname{curl} \omega$$

Invert: $$u = K(\omega)$$

K smoothing of degree 1.

We write EE:

$$\omega_t + K(\omega)\cdot\nabla\omega = 0 \quad \text{in 2D}$$
$$= \omega\cdot\nabla u \quad \text{in 3D}$$

$$\omega\cdot\nabla u = \omega_j \partial_j u = \nabla u \, \omega$$
$$(\nabla u)_{i,j} = u_{i,j} = \partial_j u_i$$

Theorem (3D vorticity transport formula)

$$\omega(\eta(y,t),t) = \underbrace{\nabla_y \eta(y,t)}_{\text{matrix}} \; \omega_0(y)$$

$\xi(y,t) = \omega(\eta(y,t),t)$ --- Lagrangian vorticity

Proof ($D_t = \partial_t + u\cdot\nabla$)

From

$$D_t \omega = \nabla u \, \omega$$

$\therefore$

$$\frac{\partial}{\partial t}\xi(y,t) = \nabla u\big|_{(\eta(y,t),t)} \; \omega(\eta(y,t),t)$$

On the other hand, $\partial_t \eta = u(\eta, t)$ implies

$\forall$ fixed vector h:

$$\frac{\partial}{\partial t}(\nabla \eta \, h) = \nabla u \, \nabla \eta \, h$$

If we choose $h = \omega_0$:

$$\nabla \eta \, h \big|_{t=0} = \nabla \eta \big|_{t=0} \omega_0 = \omega_0$$

By the uniqueness for ODEs:

$$\xi(y,t) = \nabla \eta \, h = \nabla \eta \, \omega_0 .$$