Lecture 2

Corollary : 2D $\underbrace{\omega(\eta(y,t),t)}_{\xi(y,t)} = \omega_0(y)$

(Directly: $\partial_t(\xi(y,t)) = \partial_t \omega + \partial_j \omega \underbrace{\eta_t^j}_{u_j} = 0$

Cauchy Invariance

Recall

$$v(y,t) = u(\eta(y,t),t)$$

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = 0 \\ \nabla\cdot u = 0 \end{cases}$$

Fact η is invertible as long as the sol'n is smooth. It is locally invertible $\det(\nabla\eta) = 1$ $(= J)$

Global invertibility: Hadamard's thm:
If η locally inv. & $\sup_y |(\nabla\eta)^{-1}| < \infty$
$\Rightarrow$ η is globally.

[MB, p. 142]
Berger

Let $v(y,t) = u(\eta(y,t),t)$

$$u(y,t) = v(\eta^{-1}(y,t),t)$$

Vorticity in 3D:

$$\omega = \operatorname{curl} u$$

in coordinates

$$\omega_i = \varepsilon_{ijk}\, \partial_j u_k$$

ε_{ijk} is a sign of permutation $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$

$\varepsilon_{123} = 1$

$\varepsilon_{213} = -1$

$\varepsilon_{112} = 0$

By the chain rule

$$\nabla u = \nabla v \; D(\eta^{-1}) = \nabla v \; a$$

where $\quad a = (\nabla\eta)^{-1}$

$$\xi_i = \varepsilon_{ijk}\, a_{mj}\, \partial_m v_k$$

... Lagrangian vorticity in coordinates

From the Euler eq's $\qquad \nabla p = \nabla q\, a$

$$\begin{cases} v_t + \nabla q\, a = 0 \\ \operatorname{Tr}(\nabla v\, a) = 0 \end{cases}$$

or in coordinates

$$\begin{cases} \partial_t v_i + a_{ki}\, \partial_k q = 0 \\ a_{ki}\, v_{i,k} = 0 \end{cases}$$

$v_{i,k} = \partial_k v_i$

Now by the VT formula

$$\xi = \nabla\eta \; \omega_0$$

or in coordinates

$$\varepsilon_{ijk}\, a_{mj}\, \partial_m v_k = \eta_{i,k}\, \omega_{0k}$$

Multiply with $a_{\ell i}$ & sum: $\eta_{i,k}\, a_{\ell i} = \delta_{\ell k}$

$$\varepsilon_{ijk}\, a_{\ell i}\, a_{mj}\, \partial_m v_k = \omega_{0\ell}$$

Cauchy invariance CKV.

Classical Cauchy invariance

$$\varepsilon_{ijk}\, \partial_j v_m\, \partial_k \eta_m = \omega_{0i}$$

Proof:

$$\partial_t (\varepsilon_{ijk}\, \partial_j v_m\, \partial_k \eta_m)$$

$$= \underbrace{\varepsilon_{ijk}\, \partial_j v_m\, \partial_k v_m}_{0} + \varepsilon_{ijk}\, \partial_j \partial_t v_m\, \partial_k \eta_m$$

$\varepsilon_{ijk}\, \partial_j v_m\, \partial_k v_m = \underbrace{\varepsilon_{ikj}}_{-\varepsilon_{ijk}}\, \partial_k v_m\, \partial_j v_m$

Euler

$$= -\varepsilon_{ijk}\, \partial_j (a_{\ell m}\, \partial_\ell q)\, \partial_k \eta_m$$

$$= -\varepsilon_{ijk}\, \partial_j a_{\ell m}\, \partial_\ell q\, \partial_k \eta_m - \underbrace{\varepsilon_{ijk}\, a_{\ell m}\, \partial_{j\ell} q\, \partial_k \eta_m}_{0}$$

($\partial_k \eta_m$ with $a_{\ell m}$ gives $\delta_{\ell k}$)

$(\varepsilon_{ijk}\, \partial_{jk} q)$

From $a \nabla \eta = I$

$$a_{\ell i}\, \partial_r \eta_i = \delta_{\ell r}$$

$\overset{\partial_j}{\Rightarrow}$

$$\partial_j a_{\ell i}\, \partial_r \eta_i = -a_{\ell i}\, \partial_{jr} \eta_i \qquad \Big|\; a_{rm}$$

$\Rightarrow$

$$\partial_j a_{\ell i}\, \delta_{im} = -a_{\ell i}\, \partial_{jr} \eta_i\, a_{rm}$$

$$\partial_m a_{\ell i} = - a_{\ell i} \, \partial_{jr} \eta_i \, a_{rm}$$

$\Rightarrow$

$$\partial_t (\varepsilon_{ijk} \, \partial_j v_m \, \partial_k \eta_m)$$

$$= \varepsilon_{ijk} \, a_{\ell i} \, \partial_{jr} \eta_i \, \underbrace{a_{rm} \, \partial_{\ell q} \, \partial_k \eta_m}_{\delta_{rk}}$$

$$= \varepsilon_{ijk} \, a_{\ell i} \, \partial_{jk} \eta_i = 0$$

Classical proof: The Cauchy invariance is obtained from the Weber formula:

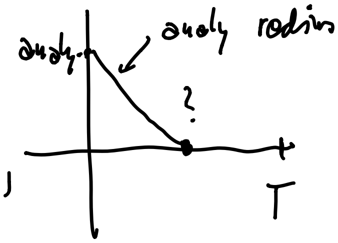
$$\partial_t (v \, \nabla \eta) = \nabla \left(\tfrac{1}{2} |v|^2 - q \right) \qquad (\partial_t v = -\nabla q \, a)$$

and then take the curl.

Regularity in Lagrangian Coordinates

For equations w/o dissipation, we expect the analytic regularity to deteriorate as t increases.

There is a persistence of analyticity (or Gevrey) for the EE. As long as the sol'n exists, it does not lose regularity.



(Bardos–Benachour, Levermore–Oliver, KV)

However, the EE in Lagrangian coordinates, the analy. radius persist (we remain in analytic space with some radius for positive time.

This fails for the Euler representation.

Anisotropic analyticity persists for all time in Lagrangian but fails in Eulerian.

Recall the EE:

$\varepsilon_{12} = 1$
$\varepsilon_{21} = -1$
$\varepsilon_{11} = \varepsilon_{22} = 0$

$$(2D)\quad \begin{cases} \varepsilon_{ij}\, a_{ki}\, \partial_k v_j = \omega_0 \\ a_{ij}\, \partial_i v_j = 0 \end{cases}$$

preservation of Lagr. vort.

$$\text{(3D)} \begin{cases} \varepsilon_{ijk}\, \partial_{li}\, \partial_{mj}\, \partial_m v_k = \omega_{0l} \\ \partial_{ij}\, \partial_i v_j = 0 \end{cases}$$

$\partial = (\nabla\eta)^{-1}$
$\eta_t = u$

This is representation of the EE w/o derivative loss.

Consider the space:

$$\|f\|_{G_{s,\delta}} = \sum_{m\geq 0} \frac{\delta^m}{m!^s} \|\partial_1^m f\|_{H^r}$$

(∂_1... anisotropic analy.)

$s \geq 1$ Gevrey class

$s = 1$ analytic class

$r > \frac{n}{2}$

$\|ab\|_{H^r} \lesssim \|a\|_{H^r} \|b\|_{H^r}$

<u>Theorem</u> (Constantin-K-Vicol) Assume $v_0 \in H^{r+1}$ where $r > \frac{d}{2}$, and assume

$$\nabla v_0 \in G_{s,\delta}$$

for some $\delta > 0$ and Gevrey index $s \geq 1$. Then $\exists\, T > 0$ and a unique sol'n

$$(v, \partial) \in C([0,T], H^{r+1}(\mathbb{R}^d)) \times C([0,T], H^r(\mathbb{R}^d))$$

s.t.

$$(\nabla v, \partial) \in L^\infty([0,T], G_{s,\delta})$$

EE this fails.

We need the Helmholtz decomposition estimate (div-curl reg.):

$$\|\nabla v\|_{L^2} \lesssim \|\operatorname{curl} v\|_{L^2} + \|\operatorname{div} v\|_{L^2}$$

which can be proven in Fourier space

$$|\xi|\,|a| \lesssim |\xi \times a| + |\xi \cdot a|$$

We also rewrite the system as

$$\operatorname{curl} v = \omega_0 - \varepsilon_{ij}\,(a_{ki} - \delta_{ki})\,\partial_k v_j$$
$$\operatorname{div} v = (a_{ij} - \delta_{ij})\,\partial_i v_j$$

<u>Proof</u> Let $m \in \mathbb{N}_0$

$$\mathcal{D}_m = \|\partial_1^m \nabla v_0\|$$

Define

$$V_m = \sup_{t \in [0,T]} \|\partial_1^m \nabla v(t)\|_{H^r}$$

$$Z_m = \sup_{t \in [0,T]} \frac{1}{t^{1/2}} \|\partial_1^m (a(t) - I)\|$$

Then

$$\|\partial_1 \nabla v\|_{H^m} \lesssim \|\partial_1^m \operatorname{curl} v\|_{H^r} + \|\partial_1^m \operatorname{div} v\|_{H^r}$$

$$\lesssim \|\partial_1^m (\omega_0 - \varepsilon_{ij}(a_{ki} - \delta_{ki})\,\partial_k v_j)\|_{H^r}$$
$$+ \|\partial_1^m ((a_{ki} - \delta_{ki})\,\partial_k v_j)\|_{H^r}$$

$$\leq \|\partial_1^m \omega_0\|_{H^r} + \|a - I\| \, \|\partial_1^m \nabla v\|_{H^r}$$
$$+ \|\partial_1^m (a - I)\|_{H^r} \, \|\nabla v\|_{H^r}$$
$$+ \sum_{j=1}^{m} \binom{m}{j} \|\partial_1^j (a - I)\|_{H^r} \, \|\partial_1^{m-j} \nabla v\|_{H^r}$$

Take a sup over $t \in [0, T]$

$$V_m \lesssim \Omega_m + T^{1/2} Z_0 V_m + T^{1/2} Z_m V_0$$
$$+ T^{1/2} \sum_{j=1}^{m-1} \binom{m}{j} Z_j V_{m-j}$$

For Z_m:

$$a \nabla \eta = I$$
$$\Rightarrow \quad \partial_t a \nabla \eta = - a \nabla \eta_t \qquad \Big| \cdot a$$
$$\partial_t a = - a \nabla v \, a$$

rewritten as

$$I - a = \int_0^t a \nabla v \, a \, d\tau$$
$$= \int_0^t (a - I) \nabla v (a - I) + \int_0^t (a - I) \nabla v$$
$$+ \int \nabla v (a - I) + \int \nabla v$$

$\Rightarrow$

$$Z_m \lesssim C T^{1/2} (T Z_0^2 V_m + \cdots)$$

For the sup

$$B_m = Z_m + V_m$$

We get

$$B_m \lesssim \Lambda_m + \overbrace{T^{1/2}\left(1 + B_0 + T^{1/2}B_0 + TB_0^2\right)}^{\text{small}} B_m$$
$$+ T^{1/2}(I + T^{1/2}) \sum_{0<j<m} \binom{m}{j} B_j B_{m-j}$$
$$+ T^{3/2} \sum_{0<|(j,k)|<m} \binom{m}{j\,k} B_j B_k B_{m-j-k}$$

Define the total analytic norm

$$N = \sum_{m \geq 0} \frac{B_m \tau^m}{m!^{(s)}}$$

We get

$$N \leq CM + CT^{1/2}N + CT^{1/2}N^2 + CT^{3/2}N^3$$

where $\ell^1 * \ell^1 \subseteq \ell^1$. Obtain

$$\boxed{N \leq 2CM}$$

if

$$CT^{1/2}\,2M + CT^{1/2}(2M)^2 + CT^{3/2}(2M)^3 \leq CM.$$