# Lecture 6

Last time: we diagonalize the linearized system, and then we used our findings for the differentiated system $(W_\alpha, Q_\alpha) \longrightarrow (W_\alpha, R)$

$$(W_\alpha, R)\begin{cases} (W_\alpha)_t + b(W_\alpha)_\alpha + \dfrac{(1+W_\alpha)R_\alpha}{1+\bar{W}_\alpha} = (1+W_\alpha)M \\ R_t + bR_\alpha = i\left(\dfrac{gW_\alpha - a}{1+W_\alpha}\right) \end{cases}$$

where $a$ is a real bilinear form

$$a := i\left(\bar{P}[\bar{R}R_\alpha] - P[R\bar{R}_\alpha]\right)$$

$$M = \frac{R_\alpha}{1+\bar{W}_\alpha} + \frac{\bar{R}_\alpha}{1+W_\alpha} - b_\alpha = \bar{P}[\bar{R}Y_\alpha - R_\alpha\bar{Y}] + P[R\bar{Y}_\alpha - \bar{R}_\alpha Y]$$

$$Y = \frac{W_\alpha}{1+W_\alpha} \quad \text{(just an auxiliary function)}$$

Obs:

1. We know this is an hyperbolic system (degenerate)
2. Dispersive. (lnww) $\begin{cases} w_t + r_\alpha = 0 \\ r_t - igw = 0 \end{cases}$ $\quad \tau = \pm\sqrt{g|\xi|}$.

Conserved energy for (lnww) $E_0(w,r) = \int \frac{1}{2}|w|^2 + \frac{1}{2i}(r\bar{r}_\alpha - \bar{r}r_\alpha)\, d\alpha$

$\mathcal{H}_0 := L^2 \times \dot{H}^{\frac{1}{2}}$ this is the space where the linear evo-lution (lnww) is well posed.

3. The energy of the linearized system is going to be

$$E^{(2)}_{lin}(w,r) = \int_{\mathbb{R}} (g+a)|w|^2 + \Im m\,(r\bar{r}_\alpha)\, d\alpha$$

$E^{(2)}_{lin}$ is a conserved energy. , $\underline{g + a \text{ is positive}}$, $= \frac{-\partial p}{\partial n} > c > 0$

S. Wu $\to$ Taylor sign conds.

Prop: Assume $\underline{R, \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{\frac{3}{2}}}$ then $a$ is nonnegative. bilinear form:

$\mathcal{H}_0 \quad \mathcal{H}_1,$

Proof: $a := i\left( P[\bar{R} R_\alpha] - P[R \bar{R}_\alpha] \right)$, $P = \frac{1}{2}(I - iH)$

$$\hat{a}(\zeta) = \int_{\xi - \eta = \zeta} \min\{\xi, \eta\} \, 1_{\{\xi, \eta < 0\}} \hat{R}(\xi) \hat{\bar{R}}(\eta) d\xi$$

$\xi, \eta$ are restricted to negative real axis, because $R$ is holomorphic,

$$\min\{\xi, \eta\} \, 1_{\{\xi, \eta < 0\}} = \int_{m > 0} \underline{1_{\{\xi < -m\}} 1_{\{\eta < -m\}} dm}$$

back in $\hat{a}$.

and invert the Fourier trans

$$a = \int | 1_{|D| > m} R |^2 dx \to \text{positive}.$$

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We return to the diff system $(W_\alpha, R)$ recall that an energy. for the full nonlinear system $(W, Q)$

$$E(W, Q) = \int_{\mathbb{R}} \frac{1}{2} |W|^2 + \frac{1}{2i} (Q \bar{Q}_\alpha - \bar{Q} Q_\alpha) - \frac{1}{4} (\bar{W}^2 W_\alpha + W^2 \bar{W}_\alpha) d\alpha,$$

$M := \operatorname{Im} \int \bar{W} Q_\alpha d\alpha.$

$(W_\alpha, R)$:

$$\|(W_\alpha, R)\|^2_{\dot{\mathcal{H}}_n} := \sum_{k=0}^{n} \| \partial^k_\alpha (W_\alpha, R) \|^2_{\underbrace{L^2 \times \dot{H}^{\frac{1}{2}}}_{\mathcal{H}_0}}, \quad n \geq 1$$

We propagate the regularity of $\underline{(W_\alpha, R)}$ and not the reg of $(W_\alpha, Q_\alpha)$.

<u>Control norm:</u> We want to obtain energy estimates, i.e to see how the energy of the sols evolves in time.

→ pointwise control norms.

→ Sobolev norms of the solution.

→ Preferable to use pointwise norms. because this eventually is going to allow you to get global result (dispersive decay estimate).

This also allows for low-reg results. (because you can use Strichatz es)

→ One can also use $L^2$-norms ($H^s$ spaces) → you do not have decay.

$$W_\alpha/(1+W_\alpha)$$

$$\text{Water Waves: } A := \|W_\alpha\|_{L^\infty} + \|Y\|_{L^\infty} + \||D|^{\frac{1}{2}}R\|_{L^\infty \cap B_2^{0,\infty}}$$

$$B := \||D|^{\frac{1}{2}} W_\alpha\|_{BMO} + \|R_\alpha\|_{BMO}$$

<u>Obs</u>: 1) A scale invariant norm, from the scaling of the problem

2). B correspond to the homogeneous $\dot{\mathcal{H}}^1$ norm of $(W_\alpha, R)$

3) Note that B controles A, Y comp is not.

$\dot{\mathcal{H}}^1$ norm control A but not the Y.

4). $\dot{H}^{\frac{1}{2}} \subset \underline{BMO}$. ($L^\infty$ missing endpoints.)

Goal: LWP theory.

Theorem: Let $n \geq 1$. The system $(W_\alpha, R)$ is well-posed for data in $\dot{\mathcal{H}}_n(\mathbb{R})$, $|W_\alpha + 1| > c > 0$. Further, the solution can be continued for as long as $A$ and $B$ remain bounded.
Periodic case = same result.

1. Nalimov, Ovsiannikov. 1968.
2. Wu 1996 LWP → in $H^s$, $s$ large.

Scaling $\dot{\mathcal{H}}^{\frac{1}{2}} \to (W_\alpha, R)$
$\dot{\mathcal{H}}^{3/2} \to (W, Q)$

$$(W(t,\alpha), Q(t,\alpha)) \to (\lambda^{-2} W(\lambda t, \lambda^2 \alpha), \lambda^{-3} Q(\lambda t, \lambda^2 \alpha))$$

3. Alazard-Burq-Zuily 2014. low reg. $s = 1 + \varepsilon$, $H^s$
↑ energy estimate
4. Hunter-I-Tataru 2015 low reg. $s = 1$, $H^s$.
↑ energy estimate.
5. Alazard-Burq-Zuily. 2018. → Strichartz estimates.
$s = 1 - \frac{1}{24} + \varepsilon$ $\left(s_{cr} = \frac{1}{2}\right)$
6. Albert Ai $s = 1 - \frac{1}{8} + \varepsilon$ 2019 Strichartz without losses.
7. Ai-I-Tataru. $\dot{\mathcal{H}}^{3/4}$ ⇒ $s = \frac{1}{2} + \frac{1}{4}$ → energy estimates
8. Ai-I-Tataru $s = \frac{1}{2} + \frac{1}{8}$ work in progress.

Recall: We diagonalized the linearized system and got

$$\begin{cases} (\partial_t + b\partial_\alpha) w + \dfrac{1}{1+\bar{W}_\alpha} r_\alpha + \dfrac{R_\alpha}{1+\bar{W}_\alpha} w = \mathcal{G}(w,r) \\ (\partial_t + b\partial_\alpha) r - i\dfrac{g+a}{1+W_\alpha} w = \mathcal{K}(w,r) \end{cases} \qquad (\ell wwi)$$

$$\mathcal{G}(w,r) = (1+W_\alpha)(P\bar{m} - \bar{P}m)$$

$$\mathcal{K}(w,r) = \bar{P}n - P\bar{n}$$

$$m = \frac{r_\alpha + R_\alpha w}{J} + \frac{\bar{R}w_\alpha}{(1+W_\alpha)^2}, \quad n = \frac{\bar{R}(r_\alpha + R_\alpha w)}{1+W_\alpha}.$$

Remarks 1) RHS terms, perturbative.

2). $(w,r)$ holomorphic functions, , but the system is an evolution in a space that is not defined now. To remedy this, apply $P$

$$P[\ell wwi] \quad \begin{cases} (\partial_t + M_b\partial_\alpha) w + P\left[\dfrac{1}{1+\bar{W}_\alpha} r_\alpha\right] + P\left[\dfrac{R_\alpha}{1+\bar{W}_\alpha} w\right] = P\mathcal{G} \\ (\partial_t + M_b\partial_\alpha) r - iP\left[\dfrac{g+a}{1+W_\alpha} w\right] = P\mathcal{K}(w,r) \end{cases}$$

$$M_b\partial_\alpha w = P[b\,\partial_\alpha w]$$

the space of the evolution $P(\ell wwi)$ happens in the space of holomorphic function.

I. We want to study LWP of $P[\ell wwi]$ in $L^2 \times \dot{H}^{\frac{1}{2}}$, because later we are going to use the energy estimates obtained here. for $(W_\alpha, R), \ldots, (W_\alpha^{(n)}, R^{(n)})$.

model system
$$\begin{cases} (\partial_t + \mathfrak{b}\partial_\alpha)w + P\left[\frac{1}{1+\bar{W}_\alpha} r_\alpha\right] + P\left[\frac{R_\alpha}{1+\bar{W}_\alpha} w\right] = G \\ (\partial_t + \mathfrak{b}\partial_\alpha)r - iP\left[\frac{g+a}{1+W_\alpha} w\right] = K \end{cases}$$

$$E^{(2)}_{lim}(w,r) = \int_{\mathbb{R}} (g+a)|w|^2 + \Im(r\bar{r}_\alpha)\, d\alpha$$

LWP in $L^2 \times \dot{H}^{\frac{1}{2}} = \dot{\mathcal{H}}_0 \to E_0$

I need to know $E^{(2)}_{lim}(w,r) \approx E_0$
$\to \|a\|_{L^\infty} < \infty$ we have.

Prop: a) our model system is well posed in $\dot{\mathcal{H}}_0$ and the following estimate holds

$$\frac{d}{dt} E^{(2)}_{lim}(w,r) = 2\operatorname{Re}\int_{\mathbb{R}} (g+a)G - i\bar{r}_\alpha K\, d\alpha$$

cubic energy est. $\to$ $+ \, O_A(\underline{AB})\, E^{(2)}_{lim}(w,r)$ $\dot{x} = x^3$ cubic energy est.
(quadratic in the control norms)

b) The linearized system $P(\mathrm{lin}\,w)$ is well posed in $\dot{\mathcal{H}}^0$

$$\frac{d}{dt} E^{(2)}_{lim}(w,r) \lesssim_A B\, E^{(2)}_{lim}(w,r)$$

quadratic energy estimate.