Lecture 7

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Model system $\begin{cases} (\partial_t + M_b \partial_\alpha) w + P\left[\frac{1}{1+\bar{W}_\alpha} r_\alpha\right] + P\left[\frac{R_\alpha}{1+\bar{W}_\alpha} w\right] = G \\ (\partial_t + M_b \partial_\alpha) r - iP\left[\frac{g+a}{1+W_\alpha} w\right] = K \end{cases}$

For the linearized system we need to replace G and K with what we previously had denoted by $\mathcal{G}$ and $\mathcal{K}$ respectively

$$\mathcal{G}(w,r) = (1+W_\alpha)(P\bar{m} - \bar{P}m)$$

$$\mathcal{K}(w,r) = \bar{P}n - P\bar{n}$$

$$m = \frac{r_\alpha + R_\alpha w}{J} + \frac{\bar{R} w_\alpha}{(1+W_\alpha)^2}, \quad n = \frac{\bar{R}(r_\alpha + R_\alpha w)}{1+W_\alpha}$$

$$E^{(2)}_{lin}(w,r) = \int_{\mathbb{R}} (g+a)|w|^2 + \Im m\,(r\bar{r}_\alpha)\,d\alpha$$

SS $\dot{\mathcal{H}}_0 = L^2 \times \dot{H}^{\frac{1}{2}}$

$A = \|W_\alpha\|_{L^\infty} + \|Y\|_{L^\infty} + \||D|^{\frac{1}{2}} R\|_{L^\infty \cap B^{0,\infty}_2}$

$B = \||D|^{\frac{1}{2}} W_\alpha\|_{BMO} + \|R_\alpha\|_{BMO}$

Prop A a) our model system is well-posed in $\dot{\mathcal{H}}_0$ and the following estimate holds:

$$\frac{d}{dt} E^{(2)}_{lin}(w,r) = 2\Re \int_{\mathbb{R}} (g+a) G - i\bar{r}_\alpha K\, d\alpha + O_A(AB) E^{(2)}_{lin}(w,r)$$

b) The linearized system $P(lnw)$ is well-posed in $L^2 \times \dot{H}^{\frac{1}{2}} = \dot{\mathcal{H}}_0$ and the following estimate holds

$$\frac{d}{dt} E^{(2)}_{lin}(w,r) \lesssim_A \underline{\underline{B}}\, E^{(2)}_{lin}(w,r)$$

$\dot{E} = E^2$

1. conformal setting / diagonalization
2. linearization!
3. LWP $\Rightarrow (W_\alpha, R) \rightarrow \frac{1}{\varepsilon} > T$ (small data)
4. Enhance lifespan of sols
5. Almost global sols + Global sol.

Plan.

Proof: • a) $\frac{d}{dt}\int (g+a)|w|^2 d\alpha = 2\,\mathrm{Re}\int (g+a)\bar{w}\,\underline{(\partial_t + M_b\partial_\alpha)w} +$

$+ a\bar{w}[b, P]w_\alpha\, d\alpha +$

$+ \int [a_t + ((g+a)b)_\alpha]|w|^2 d\alpha.$

$\frac{d}{dt}\int \mathrm{Im}(r\,\partial_\alpha \bar{r})\,d\alpha = 2\,\mathrm{Im}\int (\partial_t + M_b\partial_\alpha) r\,\partial_\alpha \bar{r}\,\partial\alpha.$

$\frac{d}{dt} E^{(2)}_{lin}(w, r) = 2\,\mathrm{Re}\int (g+a)\bar{w}\,G - i\bar{r}_\alpha K\,d\alpha + err_1$

$err_1 = \int_{\mathbb{R}} [a_t + ((g+a)b)_\alpha |w|^2\, d\alpha - 2\,\mathrm{Re}\int (g+a)\frac{R_\alpha}{1+\bar{W}_\alpha}|w|^2 d\alpha$

$-2\,\mathrm{Re}\int a\bar{w}\left([\bar{Y}, P](r_\alpha + R_\alpha w) + [P, b]w_\alpha\right)d\alpha$

Comment: we want to have commutators because this allows to move derivatives to low frequency terms

Remember: $[H, f]g_\alpha \quad [H, T_f]g_\alpha$!

$err_2 = \int_{\mathbb{R}} [a_t + b a_\alpha]|w|^2 + M(g+a)|w|^2 d\alpha - 2\,\mathrm{Re}\int a\bar{w}\left([\bar{Y}, P](r_\alpha + R_\alpha w)\right.$

$\left. + [P, b]w_\alpha\, d\alpha\right)$

B

A   Energy   Ene

$$M = \bar{P}[\bar{R} Y_\alpha - R_\alpha \bar{Y}] + P[R \bar{Y}_\alpha - \bar{R}_\alpha Y].$$

$$|err_1| \lesssim_A AB\, E^{(2)}_{lin}(w, r).$$

choose one term to show you how we estimate it.

$[\bar{Y}, P]\, r_\alpha \to$ in L^2 (CS, $a \in L^\infty$)

$$\|[\bar{Y}, P]\, r_\alpha\|_{L^2} \lesssim \underbrace{\||D|^{\frac{1}{2}} Y\|_{BMO}}_{B} \|r\|_{\dot{H}^{\frac{1}{2}}} \to \text{Coifman-Meyer commutators}$$

$Y = \frac{W_\alpha}{1 + W_\alpha}$, $\quad Y \in L^\infty$

$W_\alpha \in BMO^{\frac{1}{2}}$ (given by the control norm B).

Moser estimate: smooth function F.

$$\|F(u)\|_{BMO^{\frac{1}{2}}} \lesssim_{\|u\|_{L^\infty}} \underbrace{\|u\|_{BMO^{\frac{1}{2}}}}_{B}.$$

b) $P\mathcal{K}(w, r) \to K$

$P\mathcal{G}(w, r) \to G$

Quadratic parts + higher order

$$\begin{cases} P\mathcal{G} = \mathcal{G}^{(2)} + \mathcal{G}^{(3+)} \\ P\mathcal{K} = \mathcal{K}^{(2)} + \mathcal{K}^{(3+)} \end{cases}$$

$$\begin{cases} P\mathcal{G}^{(2)} = -P[W_\alpha \bar{r}_\alpha] + P[R \bar{W}_\alpha] \\ \qquad\quad = -[P, W_\alpha]\bar{r}_\alpha + [P, R]\bar{W}_\alpha \\ P\mathcal{K}^{(2)}(w, r) = -P[R \bar{r}_\alpha] = -[P, R]\bar{r}_\alpha \end{cases}$$

$$\underbrace{\|P\mathcal{G}^{(2)}\|_{L^2} + \|P\mathcal{K}^{(2)}\|}_{}{}_{\dot{H}^{\frac{1}{2}}} \lesssim B\left(\|w\|_{L^2} + \|r\|_{\dot{H}^{\frac{1}{2}}}\right)$$

$$\|\bar{P}[r_\alpha(1-Y)\bar{Y}\bar{W}_\alpha]\|_{L^2} \lesssim$$

$$\|[\bar{P}, (1-Y)\bar{Y}\bar{W}_\alpha]r_\alpha\|_{L^2} \overset{CM}{\leq} \underbrace{\|r\|_{\dot{H}^{\frac{1}{2}}}}_{E^{(2)}_{lin}(w,r)} \|\underbrace{(1-Y)}_{L^\infty,\ A}\underbrace{\bar{Y}}_{L^\infty,\ A}\underbrace{\bar{W}_\alpha}_{B}\|_{BMO^{\frac{1}{2}}}$$

<u>Result</u>: $L^\infty \cap BMO^{\frac{1}{2}}$ is an albera.

In the paper

$$\|uv\|_{BMO^{\frac{1}{2}}} \lesssim \|u\|_{L^\infty}\|v\|_{BMO^{\frac{1}{2}}} + \|v\|_{L^\infty}\|u\|_{BMO^{\frac{1}{2}}}$$

$\blacksquare$

<u>Higher order energy estimates</u>

The goal is to establis energy bounds for (W_α, R) and their higher derivative.

<u>Prop</u>: For any $m \geq 1$, $(\exists)$ $E^{m,(2)}$ with the following properties

(i) $E^{m,(2)}(W_\alpha, R) \approx_A E_0(\partial^{m-1}W_\alpha, \partial^{m-1}R)$

(ii) $\frac{d}{dt}E^{m,(2)}(W_\alpha, R) \lesssim_A B\, E^{m,(2)}(W_\alpha, R)$

<u>Proof</u>: $m=1 \Rightarrow$ done in part b) in prop above. (W_α, R) solved the linearized eq.

<u>$m=2$</u> $(W_{\alpha\alpha}, R_\alpha) \to$ to apply the estimates in Prop 1.

I need to modify the system $(W_{\alpha\alpha}, R_\alpha)$ so that it looks like the model eq.

$\to (W_{\alpha\alpha}, \underline{R_\alpha(1+W_\alpha)})$! (ad-hoc)

$\qquad\qquad\qquad\quad \mathbf{R}$

$\to w = e^{2\phi} W_{\alpha\alpha}$, $r = e^{2\phi}\mathbf{R}$

$$\phi = -2\,\mathrm{Re}\log(1+W_\alpha) \quad \Rightarrow \quad (w, r)\ \dot{\mathcal{H}}_0$$

Prop 1.

$n \geq 3.$ $(\partial^{n-1} W_\alpha, \partial^{n-2} R)$ Prop 2

$w = e^{\phi} W_\alpha^{(n-1)}$, $r = e^{\phi} R^{(n-1)}$ ☒

Local Well Posedness (LWP)

LWP for (W_α, R). $\to$ system you know it. , $\dot{\mathcal{H}}_n$

We know that given a solution in $\dot{\mathcal{H}}_n$ for eq (W_α, R).

That we have unif energy estimates for $n \geq 1$. $\to$ WP statement,

! $\|Y\|_{L^\infty}$ in A. - erase it.

To understand the evolution of the $\dot{\mathcal{H}}_1$ norm of the solution is convenient to f.e. language.

We say a sequence $c_k \in \ell^2$ is a $\dot{\mathcal{H}}_1$ f.e. for $(W_\alpha, R) \in \dot{\mathcal{H}}_1$ if

① $\|(W_\alpha, R)(0)\|_{\dot{\mathcal{H}}^1} \approx \sum c_k^2$

② $\dfrac{c_j}{c_k} \lesssim 2^{-\delta|j-k|}$, δ small, (3'). $\|P_k(W_\alpha, R)\|_{\dot{\mathcal{H}}_1} \leq c_k$.

Prop: System in LWP.
(W_α, R)

Proof:

Step 1. Existance of regular solution: $(W_\alpha, R)(0) \in \dot{\mathcal{H}}_n$, $n \geq 2$
and prove (∗) a sols in the same space.

$$\begin{cases} (\partial_t + P_{<H} M_{b_H} \partial_\alpha P_{<H}) W_\alpha + P_{<H} P\left[\dfrac{1+P_{<H} W_\alpha}{1+P_{<H} \bar{W}_\alpha} P_{<H} R_\alpha\right] \\ \qquad = P_{<H} G(P_{<H}(W_\alpha), P_{<H} R) \\ \ldots \end{cases}$$

$b_H = b(P_{<H} W_\alpha, P_{<H} R)$

For fixe H we have an ODE in $\dot{\mathcal{H}}_n$, $n \geq 1$,
admits a local sol (W_α^H, R^H),

$$\frac{dE^n}{dt} \lesssim C(\|(W_\alpha, R)\|_{\dot{\mathcal{H}}_n}) \|(W_\alpha, R)\|^2_{\dot{\mathcal{H}}^n}.$$

this gives a lifespan for the solution of the system above, which will depend on the size of the initial data $\|(W_\alpha, R)(0)\|_{\dot{\mathcal{H}}^n}$

$$(W_\alpha^H, R^H) \in C(\dot{\mathcal{H}}_n)$$
$$\Big\downarrow \text{weak}$$
$$(W_\alpha, R) \in L^\infty(\dot{\mathcal{H}}_n)$$

$\hookrightarrow$ strong convergence in $C(\mathcal{H}^{n-1})$ $\to$ allows we pass to the limit in the system

Step II Uniquesse... follow right away by subtracting two sols.

Step III $\mathcal{H}^1$ bounds: The solution produced above have a lifespan which depends on $\mathcal{H}^n$ size of the data. We want to prove that in fact they depend on the $\mathcal{H}^1$ size of the data:

$$\frac{d}{dt} E^n(W_\alpha, R) \lesssim AB\, E^n(W_\alpha, R)$$

$$A, B < \mathcal{H}^1.$$

$$\frac{dE^1}{dt} < C(E_1) \rightsquigarrow E^1(0).$$

$$\Rightarrow T((W_\alpha, R)(0)) \hookrightarrow \|(W_\alpha, R)(0)\|_{\mathcal{H}^1}.$$

Step IV $\mathcal{H}^1$ solutions as a limit of smooth sols

Step V Continuous dependence in $\mathcal{H}^1$. (Ifrim-Tataru follow.)