Lecture 8

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Normal forms (enhanced lifespan of solutions)

Eg 1. If one has an eq that quadratic nonlinearities

$$\frac{d}{dt} E(u) \lesssim \underbrace{\|u\|}_{\text{control norm.}} E(u)$$ → quadratic energy estimate

$\dot{x} = x^2$

$\|u(0)\| < \varepsilon$ ⇓ by Gronwall. $\overbrace{T_\varepsilon \approx \frac{1}{\varepsilon}}^{\text{quadratic lifespan}}$

Eg 2 If one performs energy estimates for an equation that has only cubic nonlinearities, and we look at small initial data,

$$\frac{d}{dt} E(u) \lesssim \|\ \|^2 E(u)$$ → cubic energy estimate

$\|u(0)\| < \varepsilon$ → by Gronwall $\underbrace{T_\varepsilon \approx \frac{1}{\varepsilon^2}}_{\text{cubic lifespan.}}$

$\dot{x} = x^3$

Obs: If one gets from eg 1 → to eg 2 then you improve the lifespan of your solution.

Obs: this idea applies to any type of nonlinearity

cubic nonlinearity → quartic or higher

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Example 1

$$\begin{cases} iu_t - au_{xx} + bu_x^2 = 0 & x \in \mathbb{R} \\ u(0,x) = u_0 \end{cases}$$

Use Hopf-Cole transformation (Evans, chp 4)

$w := \phi(u)$, $\phi: \mathbb{R} \to \mathbb{R}$ a smooth function.

$$w_t = \phi' u_t$$
$$w_x = \phi' u_x$$
$$w_{xx} = \phi'' u_x^2 + \phi' u_{xx}$$

$$
\begin{aligned}
i w_t &= i\phi' u_t \\
&= \phi' [a u_{xx} - b u_x^2] \\
&= \phi' a u_{xx} - b\phi' u_x^2 \\
&= a[w_{xx} - \phi'' u_x^2] - b\phi' u_x^2 \\
&= a w_{xx} - a\phi'' u_x^2 - b\phi' u_x^2 \\
i w_t &= a w_{xx} - u_x^2 \underbrace{[a\phi'' + b\phi']}_{=0}
\end{aligned}
$$

$$a\phi'' + b\phi' = 0.$$

$$a r^2 + b r = 0 \quad r_1 = 0,\ r_2 = -\frac{b}{a}.$$

$$w := \phi(u) = c_1 e^{u \cdot 0} + c_2 e^{u \cdot (-\frac{b}{a})} \qquad c_1 = -1,\ c_2 = 1, \Rightarrow \text{initial cond.}$$

$$w := e^{-\frac{b}{a}u} - 1$$ $\Leftarrow$ ~~Hopf-Cole transformation.~~

$$\begin{cases} i w_t - a w_{xx} = 0 \\ w_0 = e^{-\frac{b u_0}{a}} - 1. \end{cases}$$

$\Rightarrow$ solve it explicitly.

$$w(t,x) = \frac{1}{(4\pi a t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4at}} e^{-\frac{b u_0(y)}{a}} \, dy$$

bounded.

$\Rightarrow$ bounded transfunction from $H^{\frac{n}{2}+\varepsilon} \to H^{n/2+\varepsilon}$.

orig eq was quadratic $\xrightarrow{\text{bounded transf}}$ to a linear eq which has global sols

$\frac{1}{\varepsilon}$ ☺ $\longleftarrow$

Question: Do I get to always find such transfs that remove nonlinearities?

Answer: No!

Conclusion: Before computing a normal trans to remove a quadratic nonlinearity for example, one needs to solve a particular system.

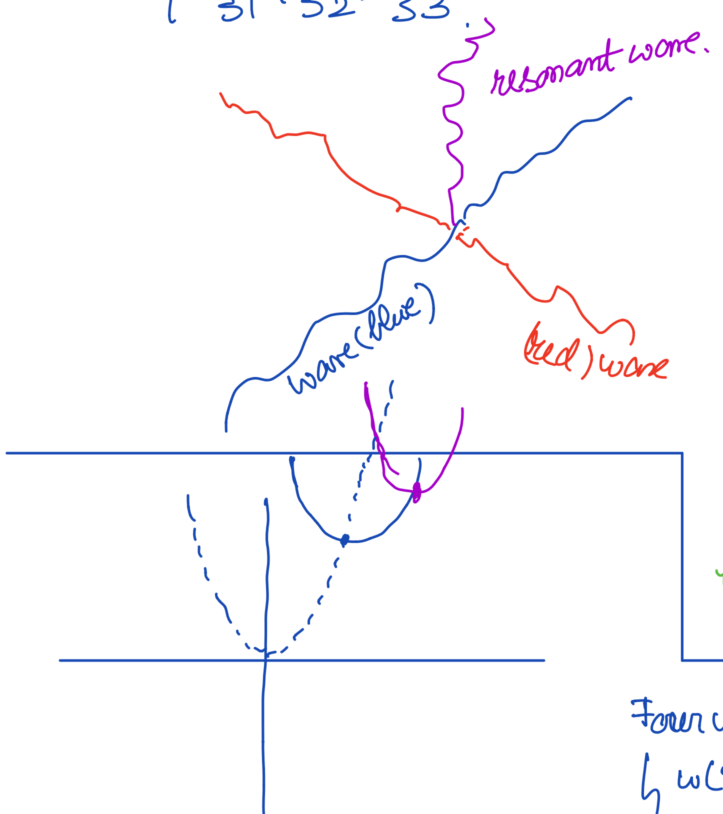
Three wave resonances system

$\omega(\xi) \to$ dispersion relation of the linear part of a given PDE.

$$\begin{cases} \omega(\xi_1) + \omega(\xi_2) = \omega(\xi_3) \\ \xi_1 + \xi_2 = \xi_3 \end{cases}$$

- Solutions means you have resonances
- no solution, no resonances (quadratic resonances) $\Leftrightarrow$ removable quadratic nonlinearity.

$\to$ if the ... of the quadratic nonlinearity vanishes on the set of resonances (pairs of resonant f) then the quadratic nonlinearity has a null form structure



Four wave resonances.

$$\begin{cases} \omega(\xi_1) + \omega(\xi_2) \pm \omega(\xi_3) = \omega(\xi_4) \\ \xi_1 + \xi_2 \pm \xi_3 = \xi_4 \end{cases}$$

History: (1) Normal form trans (NFT) Birkhoff in early 1920 ODE.
(2) Shatah 1983 Klein-Gordon eq $\to$ GWP.
(3) Simon 1983 Klein-Gordon eq $\to$ assymptotic completness.
(4) There are approaches nowadays. (see it next lectures)

Example 2: $i u_t + \sqrt{\Delta^2+1}\, u = u^2$ (half-wave for the K-G eq)

$\omega(\xi) = \sqrt{\xi^2+1}$ $\qquad x \in \mathbb{R}$

Three wave resonances system $\to \omega(\xi)+\omega(\eta) = \omega(\xi+\eta)$

$$\sqrt{\xi^2+1} + \sqrt{\eta^2+1} = \sqrt{(\xi+\eta)^2+1}$$

$$4\underbrace{(\xi^2+\eta^2-\xi\eta)}_{+} = -3$$

needs to be bdd

$$\tilde{u} := u + B(u,u) \qquad B = \text{bilinear symmetric form in } u.$$

$$u = \tilde{u} - B(u,u)$$

$$i u_t = i\tilde{u}_t - iB(u_t,u) - iB(u,u_t)$$
$$= i\tilde{u}_t - B(u^2 - \sqrt{\Delta^2+1}\,u, u) - B(u, u^2 - \sqrt{\Delta^2+1}\,u)$$
$$= i\tilde{u}_t - B(u^2,u) + B(\sqrt{\Delta^2+1}\,u, u)$$
$$- B(u,u^2) + B(u, \sqrt{\Delta^2+1}\,u)$$

$i\tilde{u}_t \neq \sqrt{\Delta^2+1}\,\tilde{u}$
$= 2B(\tilde{u}^2, u)$

$$\sqrt{\Delta^2+1}\,u = \sqrt{\Delta^2+1}\,\tilde{u} - \sqrt{\Delta^2+1}\,B(u,u)$$

$$\underbrace{i u_t + \sqrt{\Delta^2+1}\,u}_{= u^2} = i\tilde{u}_t + \sqrt{\Delta^2+1}\,\tilde{u} +$$
$$+ B(\sqrt{\Delta^2+1}\,u, u) + B(u, \sqrt{\Delta^2+1}\,u) - \sqrt{\Delta^2+1}\,B(u,u)$$
$$+ 2B(u^2,u)$$

I need to find $B(\cdot,\cdot)$ s.t.

$$u^2 = B(\sqrt{\Delta^2+1}\,u, u) + B(u, \sqrt{\Delta^2+1}\,u) - \sqrt{\Delta^2+1}\,B(u,u)$$

$$1 = \hat{B}(\xi,\eta)\sqrt{\xi^2+1} + \hat{B}(\xi,\eta)\sqrt{\eta^2+1} - \sqrt{(\xi+\eta)^2+1}\,\hat{B}(\xi,\eta)$$

$$\hat{B}(\xi,\eta) = \frac{1}{\sqrt{\xi^2+1}+\sqrt{\eta^2+1}-\sqrt{(\xi+\eta)^2+1}} = \frac{1}{\omega(\xi)+\omega(\eta)-\omega(\xi+\eta)}$$

$$B(u,u) \approx 1+\min\{|\xi|,|\eta|\}$$

$$B(u,u) = u\cdot \underset{\text{low freq}}{\underbrace{Du}} \quad \to B: H^s \to H^s.$$

H^s, $s \geq \frac{3}{2}$

$$\partial_t \tilde{u} + \sqrt{\partial^2+1}\,\tilde{u} = 2B(u^2,u)$$

$$\|B(u^2,u)\|_{H^s} \lesssim \|u^2\|_{H^s}\|u\|_{H^s} \lesssim \|\tilde{u}\|^3_{H^s} \checkmark$$

$\tilde{u}$

Obs: Both examples worked. semilinear eq!

For quasilinear problems things are harder:

Burgers Hilbert: $u_t + u u_x = Hu$ $\quad u u_x \to \frac{1}{2}uu_x + \frac{1}{2}u_x u$

$\tilde{u} = u + B(u,u)$ $\quad \frac{1}{2}i(\eta)+\frac{1}{2}i\xi$

$$\boxed{\hat{B}(\xi,\eta) = \frac{1}{2}\frac{\xi+\eta}{\operatorname{sgn}\xi+\operatorname{sgn}\eta-\operatorname{sgn}(\xi+\eta)}}$$

$\omega(\xi) \simeq -\operatorname{sgn}\xi$

$\operatorname{sgn}\xi+\operatorname{sgn}\eta-\operatorname{sgn}(\xi+\eta) = \operatorname{sgn}(\xi)\operatorname{sgn}(\eta)\operatorname{sgn}(\xi+\eta)$

~~$1+1=1$~~

~~$-1-1=-1$~~

~~$-1+1=\pm 1$~~

$B(u,u)$

We need to see the spatial formulation of B is (when possible) so the idea is to get rid of the denominator. Obviously, not in all cases this is possible. Here it is.

I $\xi,\eta>0$

II $\xi,\eta<0$

III $\xi>0$, $\eta<0$, $|\xi|>|\eta|$.

⋮

$$\widehat{B}(\xi,\eta) = \frac{1}{2}(\xi+\eta)\,\mathrm{sgn}(\xi)\,\mathrm{sgn}(\eta)\,\mathrm{sgn}(\xi+\eta)$$

$$B(u,u) = \frac{1}{2} H\big[\underbrace{Hu}_{\text{low freq}} \cdot \underbrace{Hu_x}_{\text{high freq}}\big]$$

issues !!

1. HFT is unbounded
2. the transf eq does not have energy estimate

$$\tilde{u}_t = H\tilde{u} + \mathrm{Cubic}(u,u,u)$$

$$\frac{d}{dt}\|\partial_x^k \tilde{u}\|_{L^2}^2 \lesssim \|u_x\|_{L^\infty}^2 \, \|u\|_{H^{k+1}}^2$$