# Lecture 9

Mihaela Ifrim

Normal form transformations

- semilinear examples of PDE's with quadratic nonlinearities

$\Rightarrow$ leads to bounded NFT

$$\tilde{u} = u + \underline{B(u,u)}$$

ie this bilinear form is bounded.

**Note** :

- what we did : we changed the eq under the NFT and obtained a new eq for which we were able to compute "good energy estimates", or solve the resulting eq.
  "cubic energy estimates"
- In other words we proved enhance lifespan of smooth solutions (LWP theory exists)
- small data results.

- quasilinear example Burgers-Hilbert eq

$$u_t + uu_x = Hu. \qquad x \in \mathbb{R}.$$

$$\tilde{u} = u + \underline{H[Hu \cdot Hu_x]} \to \text{unbounded transf.}$$

In the case this transf is invertible in some functional framework this is still not helping us because the resulting eq

$$\tilde{u}_t = H[\tilde{u}] + C(u,u,u)$$

does not have good energy estimates!

What can we do next?

**Obs**: NFT were used to prove enhanced lifespan of sols.
They also can be used to prove, low regularity LWP results

**Next**: Benjamin-Ono eq (IFRIM-TATARU'17)

$$\phi_t + H\phi_{xx} = \phi\phi_x . \qquad x \in \mathbb{R}.$$

$$\omega(\xi) = -\xi|\xi| . \qquad \hat{B} = \frac{\longmapsto}{\omega(\xi) + \underbrace{\omega(\eta) - \omega(\xi+\eta)}}$$

unbounded at low frequency

$$! \; \tilde{\phi} = \phi - \frac{1}{4} H\phi \cdot \partial_x^{-1}\phi - \frac{1}{4} H(\phi \cdot \partial_x^{-1}\phi) \quad \text{NFT.}$$

$$P_k^{\mp} : \qquad (\partial_t \pm i\partial_x^{\pm} \phi_k = P_k^{\pm}(\phi \cdot \phi_x)$$

$\uparrow$
projection localized at $2^k$, and ± frequencies.

$$(i\partial_t + \partial_x^2)\phi_k^+ = \underline{i P_k^+ (\phi \cdot \phi_x)}$$

issue when!
$\phi_x$ high frequency
$\phi$ low.

$$\underline{\underline{\left(i\partial_t + \partial_x^2 - i\phi_{<k} \cdot \partial_x\right)\phi_k^+}} = \boxed{\begin{aligned} & iP_k^+(\phi_{\geq k} \cdot \phi_x) \\ & + i[P_k^+, \phi_{<k}]\phi_x . \end{aligned}}$$

ALAZARD, DELORT
NFT(HH)

$$\text{Ⓘ} \quad \tilde{\phi}_k = \phi_k + B_k(\phi, \phi) \quad \text{bdd transf.}$$

gets removed by the normal form transf
we have written in red.

NFT(LH) (II) $e^{\Phi}\ldots$

TAO for the Benjamin-Ono eq

$\tilde{\Psi}_k = e^{\overset{\text{this is capital } \phi.}{\Phi}}\,\underline{\tilde{\phi}_k}$ bounded transf.

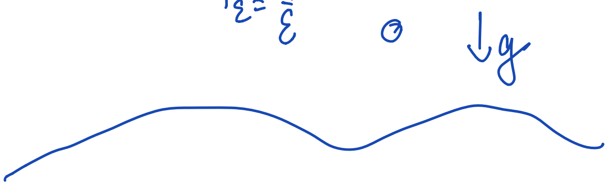
$$\begin{tikzcd}
\text{orig eq} \arrow[rr, "\text{NFT(unbounded)}"] \arrow[dr, dash, "\text{NFT(HH) bounded}"'] & & \text{transformed equation} \\
& \text{intermediate eq} \arrow[ur, "\text{NFT(HL) bounded.}"'] &
\end{tikzcd}$$

<u>Conclusion</u>: NFT can be <u>used at paradiff level.</u>

<u>Modified energy method.</u>

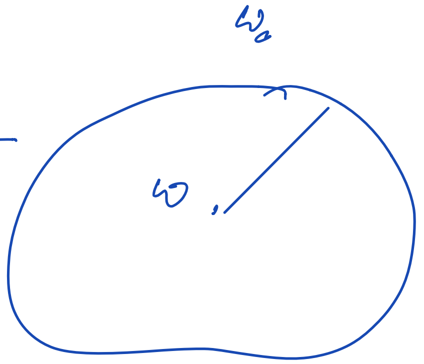
$$(BH)\begin{cases} u_t + uu_x = H[u] \longrightarrow \\ u(0,x) = u_0(x) \longleftarrow \end{cases}$$

$T_\varepsilon = \frac{1}{\varepsilon}$

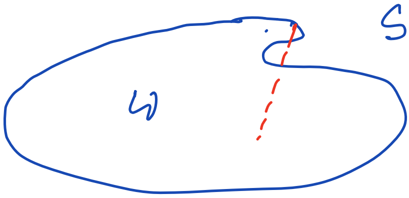


water waves

W



$\downarrow T$



Steps being a graph.

## Theorem (Hunter–Ifrim–Tataru–Wong)

(a) Let $K \geq 2$. $u_0 \in H^K(\mathbb{R})$ and $\|u_0\|_{H^2(\mathbb{R})} \lesssim \varepsilon \ll 1$

Then $(\exists)\, u \in C(I^\varepsilon; H^K(\mathbb{R})) \cap C^1(I^\varepsilon; H^{K-1}(\mathbb{R}))$, where

$$I^\varepsilon = \left[-\frac{\alpha}{\varepsilon^2}, \frac{\alpha}{\varepsilon^2}\right], \text{ where } \alpha > 0$$

$$\|u\|_{L^\infty(I^\varepsilon; H^S(\mathbb{R}))} \lesssim \|u_0\|_{H^S(\mathbb{R})} \quad \text{for } 0 \leq S \leq K$$

b). $u_1, u_2$ two solutions for (BH) then

$$\|u_1 - u_2\|_{L^\infty(I^\varepsilon; L^2(\mathbb{R}))} \lesssim \|u_1(0) - u_2(0)\|_{L^2(\mathbb{R})}.$$

**Proof**

$$v = u + H[Hu \cdot Hu_x]$$ unbounded transformation when high freq is on $Hu_x$ and low freq on $Hu$.

$$v_t = H[v] + C(u,u,u),$$

$$\frac{d}{dt}\|\partial_x^k v\|_{L^2}^2 \lesssim \underline{\|u_x\|}_{L^\infty}^2 \|u\|_{H^{k+1}}^2$$

Energy for BH: $\|u\|_{H^k}$.

$$\|\partial_x^k v\|_{L^2}^2 = \|u\|_{H^k}^2 + \underline{\underline{2\langle \partial_x^k u, \partial_x^k H[Hu \cdot Hu_x]\rangle_{L^2}}}$$

$$+ \cancel{\langle \partial_x^k H[Hu \cdot Hu_x], \partial_x^k H[Hu \cdot Hu_x]\rangle_{L^2}}$$

quartic term, and plays no role in bounding quadratic terms in the eq.

$$\approx \|u\|_{H^k}^2$$

Modiffided energy method: it is taylored to quasilinear PDEs that do not have a „large set of resonances" ≈

≈ NULL STRUCTURE of the nonlinearity + STRUCTURE of the PDE (symmetrization of the correction

I-method generalization (I-method works well for semi linear PDE's)

Colliander, Keel, Staffilani, Takaoka, Tao: a.k.a I-team.

$$E_k^{HF} = \frac{1}{2}\|\partial_x^k u\|_{L^2}^2 + \langle \partial_x^k u, \partial_x^k H[Hu \cdot Hu_x]\rangle_{L^2}$$

<u>Lemma 1:</u> $$\underset{\text{cubic}}{\underline{\underline{E_k^{HF}}}} = \underset{\text{original energy.}}{\underline{\underline{\frac{1}{2}\|\partial_x^k u\|_{L^2}^2}}} \left(1 + O\left(\|Hu_x\|_{L^\infty}\right)\right)$$

<u>Proof:</u> $$\nearrow Af := H[Hu \cdot Hf_x]$$

$$\langle f, Af\rangle_{L^2} = \langle f, H[Hu \cdot Hf_x]\rangle$$

$$= \langle -\underline{Hf}, Hu \cdot \underline{Hf_x}\rangle_{L^2}$$

move the derivative

$$= -\frac{1}{2}\int_{\mathbb{R}} \underline{\underline{Hu_x}}\, |Hf|^2 dx = O(\|Hu_x\|_{L^2})\|f\|_{L^2}^2$$

$$\partial_x^k H[Hu \cdot Hu_x] = A \underbrace{\partial_x^k u}_{f} + l.o.t.$$

$$H\left[\partial_x^j H\underline{u_x} \cdot \partial_x^{k-1-j} Hu_x\right], \quad 0 \le j \le k-1$$

$$\|\partial_x^j Hu_x \cdot \partial_x^{k-j-1} Hu_x\|_{L^2} \lesssim \|\partial_x^j Hu_x\|_{L^p} \|\partial_x^{k-1-j} Hu_x\|_{L^q}$$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

Gagliardo-Nirenberg ineq.

$$1 \le r, q \le \infty, \quad \frac{j}{m} \le \alpha < 1, \quad \mathbb{R}^n$$

$$\|\partial^j u\|_{L^p} \lesssim \|\partial^m u\|_{L^r}^{\alpha} \|u\|_{L^q}^{1-\alpha}$$

Scaling constraint

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1-\alpha}{q}$$

$$\| \partial_x^j \underbrace{Hu_x}\|_{L^p} \lesssim \| Hu_x \|_{L^\infty}^{1-\alpha} \| \partial_x^{k-1} Hu_x \|_{L^2}^{\alpha}$$

For $\|\dots\|_{L^2}$ same interpolation for the second term

$$\Rightarrow \| \partial_x^j Hu_x \cdot \partial_x^{k-1-j} Hu_x \|_{L^2} \lesssim \| Hu_x \|_{L^\infty} \overbrace{\| \partial_x^{k-1} u_x \|_{L^2}}^{\|u\|_{H^k}}$$

<u>Lemma 2</u>: The $E_k^{NF}$ satisfies the following energy estimate:

$$\frac{d}{dt} E_k^{HF} \lesssim \underline{\| Hu_x \|_{L^\infty}^2} \; \|u\|_{H^k}^2$$

Control norm is $\| Hu_x \|_{L^\infty}$

<u>Proof</u>: $$-\frac{d}{dt} E_k^{HF}(u) = \int_{\mathbb{R}} \partial_x (uu_x) \partial_x^k H[Hu \cdot Hu_x]\, dx$$

$$+ \int \partial_x^k u \left\{ \partial_x^k H[H[uu_x] \cdot Hu_x] + \partial_x^k H[Hu \cdot H[uu_x]]_x \right\} dx.$$

$$= \int_{\mathbb{R}} -\partial_x^k H[uu_x] \partial_x^k [Hu \cdot Hu_x] + \partial_x^k Hu_x \cdot \partial_x^k [Hu \cdot H(uu_x)]\, dx$$

a general term of the energy is $\longrightarrow$

$$\boxed{\begin{array}{c} H\partial^\alpha u \; H\partial^\beta u \; H(\partial^\gamma u \, \partial^\delta u) \\ \alpha + \beta + \gamma + \delta = 2k+2. \end{array}}$$

As long as $1 \le \alpha, \beta, \gamma, \delta \le k$, $\alpha + \beta \ge 3$, $\delta + \gamma \ge 3$, we are fine and we call <u>them good terms.</u>

meaning we can bound them with the energy and the control norm.