Lecture 10

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Last time: Burgers Hilbert eq

$$\begin{cases} u_t + uu_x = Hu \\ u(0,x) = u_0 \end{cases}$$

- We constructed the NFT and the modified energy functional

$$v = u + H[Hu \cdot Hu_x]$$

$$E_k^{NFT}(u) = \| \partial_x^k u \|_{L^2} + 2 \langle \partial_x^k u, \partial_x^k [H[Hu \cdot Hu_x]] \rangle$$

- We showed it is equivalent ($E_k^{NFT} \approx E_k = \|u\|_{H^k}$).
- We started the energy estimate and we got to the following step.

$$-\frac{d}{dt} E_k^{NFT}(u) = \int_{\mathbb{R}} -\partial_x^k H(uu_x)\, \partial_x^k [Hu \cdot Hu_x] + \int_{\mathbb{R}} \partial_x^k Hu_x\, \partial_x^k [Hu \cdot H(uu_x)]\, dx.$$

Here we looked at „a general term" in the expression on the right hand side

$$H\partial^\alpha u \; H\partial^\beta u \; H(\partial^\gamma u \; \partial^\delta u), \text{ where } \alpha+\beta+\gamma+\delta = 2k+2$$

and concluded that as among these terms the easy ones to estimate and we call them „good terms" are the ones where $1 \le \alpha, \beta, \gamma, \delta \le k$, $\alpha + \beta \ge 3$, $\delta + \gamma \ge 3$

good terms = you can Holder and then interpolation as we did in the „equivalence of norms" Lemma!

- If the second operator in each of the exp^g apply to the second factor i.e.

$$0=\int -\partial_x^k H(uu_x)\, Hu\cdot \partial_x^k Hu_x + \int \partial_x^k H u_x\, Hu\cdot \partial_x^k H(uu_x)$$

- We now fully expand the derivative and take a general term

$$I_\alpha = \int_{\mathbb{R}} -\partial_x^k H(uu_x)\cdot \partial_x^\alpha Hu \cdot \partial_x^{k-\alpha} Hu_x + \partial_x^k Hu_x\cdot \partial_x^\alpha Hu \cdot \partial_x^{k-\alpha} H(uu_x)\,dx$$

$$1\le \alpha \le k$$

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+good.

$$1\le \alpha \le k$$

if we distribute these derivatives and at least one falls onto u we are good.! Same form as terms we called good!

Lets look at what happens when this is not the case

$$I_\alpha = \int_{\mathbb{R}} H[u\ \partial_x^k u_x]\, \partial_x^\alpha Hu\cdot \partial_x^{k-\alpha} Hu_x + \int_{\mathbb{R}} \partial_x^k Hu_x\cdot \partial_x^\alpha Hu\ H[u\partial_x^{k-\alpha} u_x)\,dx$$

commute u! fine $=0$

$$=\int -\underbrace{[H,u]\partial_x^k}_{\partial_x} u_x\cdot \partial_x^\alpha Hu\cdot \partial_x^{k-\alpha} Hu_x + \partial_x^k Hu_x\cdot \partial_x^\alpha Hu\cdot \underbrace{[H,u]\partial_x^{k-\alpha}}_{d}\,dx$$

$$\|[H,u]\,\partial_x^k u_x\|_{L^2} \lesssim \|u_x\|_{L^\infty}\,\|\partial_x^k u\|_{L^2}$$

$$\frac{d}{dt}E^{NFT} \lesssim \|u_x\|_{L^\infty}^2 E^{NFT} \longrightarrow \text{cubic lifespan of solution}$$

✓

Water Waves

$$\begin{cases} W_t + F(1+W_\alpha) = 0 \\ Q_\alpha - igW + FQ_\alpha + P\left[\frac{|Q_\alpha|^2}{J}\right] = 0 \end{cases}$$

expand everything and retain only the linear and quadratic part.

$\gamma = \sqrt{g|\xi|}$

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{|1+W_\alpha|^2}\right]$$

$$F \approx Q_\alpha - W_\alpha Q_\alpha + P[\bar{Q}_\alpha W_\alpha - Q_\alpha \bar{W}_\alpha]$$

$$\begin{cases} W_t + Q_\alpha = G^{(2)} + \cancel{G^{(3+)}} \longrightarrow G^{(2)} = -P[\bar{Q}_\alpha W_\alpha - Q_\alpha \bar{W}_\alpha] \\ Q_t - igW = K^{(2)} + \cancel{K^{(3+)}} \longrightarrow K^{(2)} = -Q_\alpha^2 - P[|Q_\alpha|^2] \end{cases}$$

$$\begin{cases} \tilde{W} = W + \underbrace{W_{[2]}} \\ \tilde{Q} = Q + \underbrace{Q_{[2]}} \end{cases}$$

sum of bilinear terms (W, Q)

$W_{[2]}$ and $Q_{[2]}$ = ?

- holomorphic components
- mixed terms.

$W_{[2]}$: (1) Holomorphic part

$$B^h(W,W) + C^h(Q,Q) + D^h(W,Q)$$

(2) Antiholomorphic part

$$B^a(W,\bar{W}) + C^a(Q,\bar{Q}) + D^a(Q,\bar{W}) + E^a(\bar{Q}, W)$$

$$W_{[2]} = B^h + C^h + D^h + B^a + C^a + D^a + E^a$$

$\underline{\underline{Q_{[2]}}}$ ① Holomorphic part.

$$A^h(W, Q)$$

② Antiholomorphic part.

$$A^a(W, \bar{Q}) + G^a(Q, \bar{W}).$$

$$Q_{[2]} = A^h + A^a + G^a$$

Holomorphic
$$\begin{cases} W_t + Q_\alpha = 0 \\ Q_t - igW = -Q_\alpha^2 \end{cases}$$

Antiholomorphic
$$\begin{cases} W_t + Q_\alpha = G^{(2)} \\ Q_t - igW = P[|Q_\alpha|^2] \end{cases}$$

$$\boxed{\begin{cases} \tilde{W} = W - 2P[\mathrm{Re}\, W\, W_\alpha] \\ \tilde{Q} = Q - 2P[\mathrm{Re}\, W\, \underset{\uparrow R}{Q_\alpha}] \end{cases} \qquad R = \frac{Q_\alpha}{1+W_\alpha}}$$

NFT for 2D water water waves, and it is unbounded.

Now, we return to the linearized eq

$$\begin{cases} w_t + P[b w_\alpha] + P\left[\frac{1}{1+\bar{W}_\alpha} r_\alpha\right] + P\left[\frac{R_\alpha}{1+\bar{W}_\alpha} w\right] = -P[W_\alpha \bar{r}_\alpha - R \bar{w}_\alpha] \\ r_t + P[b r_\alpha] - iP\left[\frac{g+a}{1+W_\alpha} w\right] = -P[R \bar{r}_\alpha] + G + K \end{cases}$$

<u>Model system + quadratic terms.</u>

We want to show we can construct a modified energy for this system.

Linearize the NFT we have for water waves.

$$\begin{cases} \tilde{w} = w - 2P[\mathrm{Re}\, w \cdot W_\alpha] - 2P[\mathrm{Re}\, W \cdot w_\alpha] \\ \tilde{q} = q - 2P[\mathrm{Re}\, v \cdot R] - 2P[\mathrm{Re}\, W \cdot r_\alpha] \end{cases}$$

$r = q - Rw$

$L^2 \times \dot{H}^{\frac{1}{2}}$

(quadratic + cubic terms) $\|\tilde{w}\|_{L^2}$ $\|\tilde{q}\|_{\dot{H}^{\frac{1}{2}}}$ (quadratic + cubic terms)

$\|v\|_{H^s} \to$ expanded kept the quadratic + cubic terms!

Recall

$\langle u,v\rangle_{L^2} = \int_{\mathbb{R}} \bar{u}\, v \, d\alpha$

$\langle u, Pu\rangle_{L^2} = \int \bar{u}\, Pu \, d\alpha$

$= \int \overline{\Phi \bar{u}}\, u \, d\alpha = \int \bar{u}\, u \, d\alpha$

$\|\tilde{q}\|_{\dot{H}^{\frac{1}{2}}} = \mathrm{Re} \int_{\mathbb{R}} \overline{|D|^{\frac{1}{2}} r}\, |D|^{\frac{1}{2}} Rw \ldots$ move the half derivative

cubic terms

$= \mathrm{Re} \int -i \bar{r}_\alpha \cancel{P}[\bar{w} R] - i \bar{r}_\alpha\, 2\cancel{P}[\mathrm{Re}\, W\, r_\alpha]\, d\alpha$ $\quad \Phi \quad \Phi$

$\mathrm{Re} \int -i \bar{r}_\alpha \bar{w} R \, dx$

$= \mathrm{Im} \int \bar{R} w\, r_\alpha \, dx. \quad \ldots$ same $\|\tilde{w}\|_{L^2} \ldots$

$$E^{(3)}_{lin}(w,r) = \int_{\mathbb{R}} (g+a)|w|^2 + \mathrm{Im}(r\bar{r}_\alpha) + 2\,\mathrm{Im}(\bar{R} w\, r_\alpha) - 2R(\bar{W}_\alpha w^2)\, d\alpha.$$

the quadratic part of this energy is something we already had!

The green underlined terms (cubic ones) come from the normal form analysis!

Corrections that come from the normal form computations!

$\to \frac{d}{dt} E^{(3)}_{lin}(w,r) \lesssim_A AB\, E^{(3)}_{lin}$ ✓

this estimate leads to $T_\varepsilon = \frac{1}{\varepsilon^2}$ which is what we call a cubic lifespan.

Energy: assume we start with the energy suggested by the linear

$\int g \| w \|^2 + \frac{1}{2i} (u \bar{u}_\alpha + c.c)\, dx +$ cubic equation and we try to correct this with the cubic corrections from the normal form but this will not be enough!

a " $\leftarrow$ quasilinear correction (additional to NF corrections!)

There we need " a correction

Method to construct the modified energy

Step 1: $E_k^{NF,3}(u) = (\text{quadratic} + \text{cubic terms})\, E_k^{lin}(\cdot\,\tilde{u})$

Step 2: When doing the energy estimates using your $E_k^{NF,3}$, you observe you cannot close the energy estimate (sometimes you can eg Burgers-Hilbert eq.)

$$E_k^{NF,3}(u) = \underline{\underline{E_{k,high}^{NF,3}}}(u) + E_{k,low}^{NF,3}(u)$$

Step 3

modify this part so that it sees the quasilinear character of the problem.

match this at cubic order to the quasilinear energy

$$= \boxed{E_k^{ql}(u) + \text{quartic.}}$$

So now we can define the "modified energy" as

$$E_{modified}^{k,(3)} := E_k^{ql}(u) + E_{k,low}^{NF,3}(u)$$

This method gives the cubic lifespan! ↗ result of Hunter-Ifrim-Tataru

Theorem: Let $\varepsilon \ll 1$, assume the initial data for eq (W_α, R) on $\mathbb{R}$ on $\mathbb{T}^1$, satisfies $\| W_\alpha(0), R(0)\|_{\dot{\mathcal{H}}^1} \le \varepsilon$

Then the sols (3) on ε^{-2} time scale and, and it satisfies a similar bound. Higher regularity result also follow!!

Remarks

1 gravity ww

1) same regularity used to prove cubic results, as the reg in LWP
2) find the right variables (diagonal variable)
3) on having a normal form.
4) different approach than other works, which do not get results in the same regularity setting as the LWP theory (the cubic lifespan result are in high regularity setting)

2 Capillary w.w. → inf botton, surf. tension, no gravity, Ifrim-Tataru

3 Constanst vorticity WW → inf depth gravity, Ifrim-Tataru

4 Shallow water → finite depth, g, no surface tension.
Harrop-Griffiths-Ifrim-Tataru

upcoming results:
Ifrim-Koch-Tataru

~ Finish ~