Summer school lecture 6

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Local well-posedness for quasilinear evolutions (continued).

General problem:

$$u_t = N(u) \quad , \qquad u(0) = u_0 \in H^s$$

Model case: [symmetric hyperbolic system]

$$u_t = A^j(u)\, \partial_j u$$

Wellposedness [Hadamard]

1. existence
2. uniqueness
3. continuous dependence

Linearized eqn:

$$v_t = DN(u)\, v$$

Paradifferential equation:

$$w_t = T_{DN(u)}\, w$$

Energy estimates

$s \geq 0, \qquad E^s(u) \approx \|u\|_{H^s}^2$

$$\text{(EE)}: \quad \frac{d}{dt} E^s(u) \lesssim_A B \|u\|_{H^s}^2$$

Model problem: $A = \|u\|_{L^\infty}, \quad B = \|\nabla u\|_{L^\infty}$

Energy estimates for the linearized eqn:

$$v_t = DN(u)\, v$$

$$v_t = T_{DN(u)} v + F^{lin}(u)\, v$$

For model problem:

$$v_t = A^j(u)\, \partial_j v + DA^j(u) v\, \partial_j u$$

$$v_t = T_{A^j(u)} \partial_j v + \dots \underbrace{T_{DA^j(u) v}\, \partial_j u}$$

$$\text{(EE-lin)} \quad \frac{d}{dt} \|v\|_{L^2}^2 \lesssim_A B \|v\|_{L^2}^2$$

$\Downarrow$

Can move it to H^σ only for $|\sigma| \leq s-1$.

This is because of the red term above.

Energy estimates for differences

For the model problem:

$u_1, \quad u_2, \qquad v = u_1 - u_2$

$$v_t = A^j(u_1)\,\partial_j v + \underbrace{(A^j(u_1) - A^j(u_2))}_{v}\partial_j u_2$$

Same estimate as (EE)-lin,

$$A = A_1 + A_2 \quad , \quad B = B_1 + B_2$$

<u>Corollary</u> Uniqueness !

(by Gronwall)

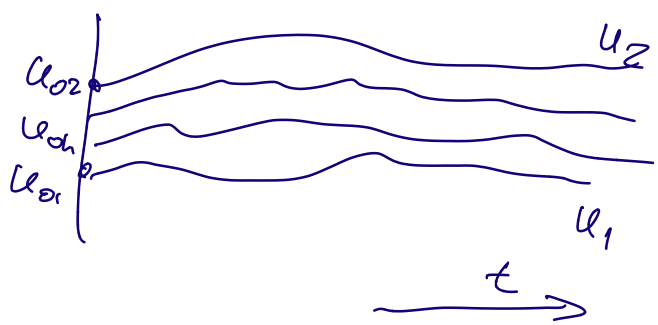
<u>Remark</u> Linearized eqn EE $\Rightarrow$ Difference Eds.

$\underbrace{u_{01}, \; u_{02}}$

$u_{0h}, \quad h \in [1,2]$

$\Downarrow$

u_h



$$u_2(t) - u_1(t) = \int_0^1 \underbrace{\frac{d}{dh} u_h(t)}_{\text{solutions for linearized equation.}} \, dh$$

Next: Existence of solutions

Method 1 $\Rightarrow$ Iterative construction

$\to$ $u^{(0)} = u_0$

$\to$ Construct $u^{(n)}$ approx. sln's

$\to$ $u^n \to u$

$\partial_t u^{n+1} = N(u^n)$ $\to$ loose derivatives

$\partial_t(u^{n+1} - u^n) = DN(u^n)(u^{n+1} - u^n)$

$\to$ Nash – Moser scheme

still looses derivatives

$$u_t = T_{DN(u)}\, u + F(u)$$

$$u^{n+1}_t = T_{DN(u^n)}\, u^{n+1} + F(u^n), \quad u^{n+1}(0) = u_0$$

Two steps:

(a) Prove uniform bounds for $u^{(n)}$ in H^s.

(b) Prove convergence in L^2.

ⓐ Uniform bounds :

→ Paradiff eqn. is well-posed in all Sobolev spaces ✓

→ $F : H^s \to H^s$

ⓑ Convergence via L^2 bounds :

$$\boxed{(u^{n+1}-u^n)_t = T_{DN(u^n)}(u^{n+1}-u^n)} + \boxed{F(u^n) - F(u^{n-1})} + T_{DN(u^n)-DN(u^{n-1})}\, u^{n-1}$$

✓ paradiff eqn. L^2 well-posed

→ F is Lip

→ more delicate

$$\frac{d}{dt}\|u^{n+1}-u^n\|_{L^2} \leq C\|u^{n+1}-u^n\|_{L^2} + \|u^n-u^{n-1}\|_{L^2}$$

Gronwall in $[0,T]$

$$\|u^{n+1}-u^n\|_{L^\infty[0,T;L^2]} \leq C\int_0^T \|u^n - u^{n-1}\|_{L^2}$$

$$\leq C \cdot T \cdot \|u^n-u^{n-1}\|_{L^\infty[0,T;L^2)}$$

↓ Large but fixed

$$\leq \gamma \|u^n - u^{n-1}\|_{L^\infty L^2}$$

<1 for $T \ll 1$.

Conclusion: u^ε bdd in $L^\infty H^s$

$u^\varepsilon \to u$ in $L^\infty L^2$

$\to$ $u^\varepsilon \to u$ in $L^\infty H^\sigma$, $0 \le \sigma < s$

$\to$ $u \in L^\infty H^s$

$u \in C H^s$ missing

Method 2.

$\boxed{\text{Time discretization idea}}$

$$\begin{array}{ccccc} u_0 & \longrightarrow u(\varepsilon) & \longrightarrow & u(2\varepsilon) & \longrightarrow \cdots \quad \cdots \\ t=0 & t=\varepsilon & & t=2\varepsilon & \end{array}$$

$\to$ Energy bound:

$$E^s(u((k+1)\varepsilon)) \le (1+C\varepsilon) E^s(u(k\varepsilon))$$

$\to$ Approximate solutions:

$$u((k+1)\varepsilon) = u(k\varepsilon) + \varepsilon N(u(k\varepsilon)) + O(\varepsilon^2)$$

Can use a weak topology (e.g. L^2)

Only one step is needed.

$$u_0 \longrightarrow u_1$$

$\longrightarrow$ $E^s(u_1) \le (1+C\varepsilon) E^s(u_0)$

$\longrightarrow$ $u_1 = u_0 + \varepsilon N(u_0) + O(\varepsilon^2)$

Two step idea:

(1) Regularize $u_0 \to \tilde{u}$

(2) Set $u_1 = \tilde{u} + \varepsilon N(\tilde{u})$

(1) $E^s(\tilde{u}) \leq E^s(u_0)(1 + C\varepsilon)$

Model problem:

$$\tilde{u} = P_{\leq \varepsilon^{-2}} u_0$$

$$\|\tilde{u}\|_{H^s} \leq \|u_0\|_{H^s}$$

$$\|\tilde{u} - u_0\|_{L^2} \leq \varepsilon^N$$

(2) Show that energy does not increase much from $\tilde{u}$ to u_1

$\to$ like moving energy est:

$$\|u_1\|_{H^s}^2 = \|\tilde{u}\|_{H^s}^2 + 2\varepsilon\langle \tilde{u}, N(\tilde{u})\rangle_{H^s}$$

$$+ \varepsilon^2 \underbrace{\|N(\tilde{u})\|_{H^s}^2}$$

estimate using the reg.

integrate by parts, as in en. est.

Other methods: approximate the eqn:

$$u_t = N(u) - \varepsilon(-\Delta)^k u$$

Next: Approximate rough solutions by smooth solutions.

$$\begin{array}{ccc} u_0 & \longrightarrow & u \\ \uparrow & & \uparrow \\ u_0^h & \longrightarrow & u^h \end{array}$$

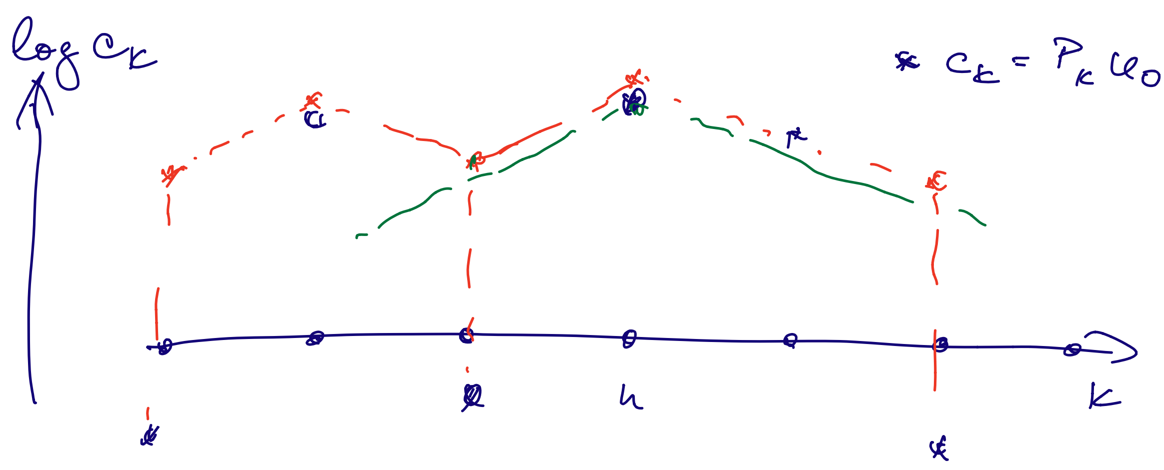
$h \in \mathbb{N}$ dyadic scale ($\leq 2^h$)

<u>Frequency envelopes:</u>

c_k frequency envelope for u_0

$\rightarrow \ \| P_k u_0 \| \leq c_k$

$\rightarrow \ c_k$ is slowly varying.



Bounds for u_0^h:

$$\| u_0^h \|_{H^s} \leq \| u_0 \|_{H^s}$$

$$\begin{cases} \| u_0^h \|_{H^{s+k}} \leq c_h\, 2^{kh} \| u_0 \|_{H^s} \\ \| u_0^{h+1} - u_0^h \|_{L^2} \leq c_h \cdot 2^{-sh} \| u_0 \|_{H^s} \end{cases}$$

Propagate in time via EE.

$$\begin{cases} \| u^h \|_{H^{s+k}} \leq c_h\, 2^{kh} \| u_0 \|_{H^s} \\ \| u^{h+1} - u^h \|_{L^2} \leq c_h \cdot 2^{-sh} \| u_0 \|_{H^s} \end{cases}$$

→ convergence in L^2 $u^h \to u$ in L^2

→ Look at $u^{h+1} - u^h$:

$$\begin{cases} \text{(a) } L^2 \text{ bound} \\ \text{(b) } \| u^h - u^{h+1} \|_{H^{s+k}} \leq \underbrace{c_h}\, 2^{kh} \| u_0 \|_{H^s} \end{cases} \qquad k > 0$$

Interpolating between L^2 and H^{s+k}, we obtain the same bound in all H^σ spaces, for $0 < \sigma < \infty$.

Compute differences:

$$\| u_k - u \|_{H^s}^2 \le \sum_{\ell \ge k} \| u_{\ell+1} - u_\ell \|_{H^s}^2$$
$$\le \sum_{\ell \ge k} c_\ell^2$$
$$:= c_{\ge k}^2 \to 0 \text{ as } k \to \infty$$

Continuous dependence on data.

$$\begin{array}{cccl} \cdot\ u_{0k} & \longrightarrow & u_0 & \text{in } H^s \\ \Downarrow & & \Downarrow & \\ \text{sln.}\quad u_k & \xrightarrow{??} & u & \text{in } C(0,T;H^s) \end{array}$$

Use regularization.

$$\begin{array}{cccl} u_{0k}^h & \longrightarrow & u_0^h & \to \text{ in } H^\sigma, \sigma \ge 0 \\ \Downarrow & & & \\ u_k^h & \longrightarrow & u^h & \to \text{ in } H^\sigma\ \sigma \ge 0. \end{array}$$

$$\| u_k - u \|_{CH^s} \le \| u_k - u_k^h \| + \| u - u^h \|$$
$$+ \underbrace{\| u_k^h - u^h \|}_{\Downarrow\ 0}$$

$$\limsup_{k\to\infty} \|u_k - u\|_{CH^s} \le \limsup \|u_k - u_k^h\| + \|u - u^h\|$$

Use frequency envelopes.

$$\left.\begin{array}{ccc} u_k & \longrightarrow \text{FE} & c_{kj} \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ u & \longrightarrow \text{FE} & c_j \end{array}\right\} \text{convergence in } L^2.$$

$$\limsup_{k\to\infty} \|u_k - u\|_{CH^s} \le \limsup_{k\to\infty} c_{k,\ge h} + c_{\ge h}$$

$$= 2\, c_{\ge h}$$

Let $h \to \infty$, RHS $\to 0 \implies$

$$\lim_{k\to\infty} \|u_k - u\|_{CH^s} = 0$$