Summer School Lecture 7

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Q: Given a quasilinear PDE, what can we say about the lifespan of the solutions?

<u>Q1</u>. Suppose the initial data is small,

$$\|u_0\| \leq \varepsilon$$

$$T(u_0) \geq T_\varepsilon$$

ⓐ $u_0 \in H^s$, small $\to$ Mihaela

ⓑ $u_0 \in H^s$, small + localized $\to$ Daniel

The vector field method.

Energy estimates:

$$E^s(u) \Rightarrow H^s \text{ energy}$$

$$\frac{d}{dt} E^s(u) \le B \; E^s(u)$$

$\Downarrow$

L^∞ type control norm.

[may decay]

- prior discussion:

L^∞ norms have dispersive decay in linear flows

→ via fundamental solutions

→ stationary phase method

Nonlinear flow: No access to such methods

→ we control energies ⇒ H^s norms

→ useless for pointwise decay

→ can we get access to more energies which can give pointwise decay?

$\boxed{\text{Vector field method}}$

A brief history:

- nonlinear wave equation

  60's → Morawetz
  actual vector fields from wave equ. symmetries

  80's → John

  90's → Klainerman, Christodoulou

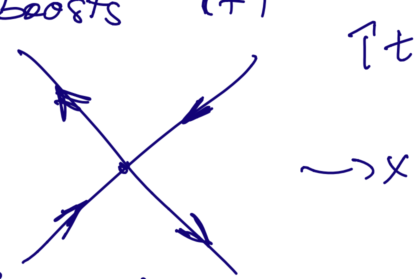
  90's → Alinhac

  90's → Hormander, Sogge books.

<u>Wave equation</u>

$$\Box u = 0, \qquad \Box = \partial_t - \Delta_x$$

Lorentz group
→ spatial rotations
→ Lorentz boosts (+)



Generators for the Lorentz group:

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i \qquad \Big| \qquad S = t\partial_t + x\partial_x$$

$$\Omega_{0j} = x_i \partial_t + t\,\partial_{x_i}$$

$$\{\Omega_{ij}, \Box\} = 0$$

If $\Box u = 0 \Rightarrow \Box \Omega u = 0$

$E(u)$ = conserved $\to$ $E(\Omega u)$ conserved.

Q: $$\|\nabla u\|^2_{L^\infty} \leq t^{-\frac{d-1}{2}} \sum_\alpha E(\Omega^\alpha u)$$

Klainerman - Sobolev inequalities

$\to$ They are Sobolev embeddings in a good frame of reference.

$\to$ in our work
- Metcalfe - Tataru - T.
- Ifrim - Stingo

What about nonlinear waves?

Suppose Ω generates a symmetry for nonlin wave equ. for u.

Ωu = sln. for linearized equ.

$\Omega^\alpha u$ = linearized equ. with source terms

Linear model equations

Example 1: The Schrodinger equation.

$$(i\partial_t - \Delta) u = 0 \quad , \quad u(0) = u_0$$

Dispersive decay:

$$\| u(t) \|_{L^\infty} \leq t^{-\frac{n}{2}} \| u(0) \|_{L^1}$$

Challenge: Prove $t^{-\frac{n}{2}}$ decay if $u(0) \in L^2$ is localised.

Vector fields $\Rightarrow$ Galilean symmetry

$u_0 \in L^2$ localized: $u_0 \in L^2$

$x \cdot u_0 \in L^2$

$\vdots$

$x^\alpha u_0 \in L^2$

$$\begin{tikzcd}
u_0 \arrow[r, "e^{it\Delta}"] \arrow[d] & u(t) \arrow[d] \\
x \cdot u_0 \arrow[r, "e^{it\Delta}"] & \underbrace{?}_{Lu \;\to\; L = e^{it\Delta} x e^{-it\Delta}}
\end{tikzcd}$$

Egorov theorem in microlocal analysis

$$L_j = x_j - 2it\,\partial_j \quad \Rightarrow$$

$$[L_j, i\partial_t - \Delta] = 0$$

$u_0 \in L^2$, localized $\Rightarrow$

$$\| L^\alpha u(t) \|_{L^2} = \| x^\alpha u_0 \|_{L^2} < \infty .$$

Claim:

$$\| u(t) \|_{L^\infty} \leq t^{-\frac{n}{2}} \cdot \sum_{\alpha \leq N} \| L^\alpha u \|_{L^2}$$

Interpret as a Sobolev embedding:

$$v = e^{i \frac{x^2}{4t}} u$$

$$e^{i \frac{x^2}{4t}} L u = 2t \cdot \partial v$$

$$\rightarrow \quad \| v \|_{L^\infty} \leq t^{-\frac{n}{2}} \sum_{|\alpha| \leq N} t^{|\alpha|} \| \partial^\alpha v \|_{L^2}$$

$$\| v \|_{L^\infty} \leq \| v \|_{\dot H^{\frac{n}{2}}}$$

False but almost true!

$$\| v \|_{L^\infty}^2 \leq \| v \|_{\dot H^{\frac{n}{2}-1}} \cdot \| v \|_{\dot H^{\frac{n}{2}+1}}$$

True!

Interpolation inequality

Example 2 KdV equation, linear

$$u_t + u_{xxx} = 0$$

Vector field: $L = x - 3t\partial_x^2$ // better

Scaling symmetry $\tilde{L} = \partial x - 3t\partial_x^3$

$\|u\|_{L^2} + \|Lu\|_{L^2} \geq L^\infty$ bounds for u.

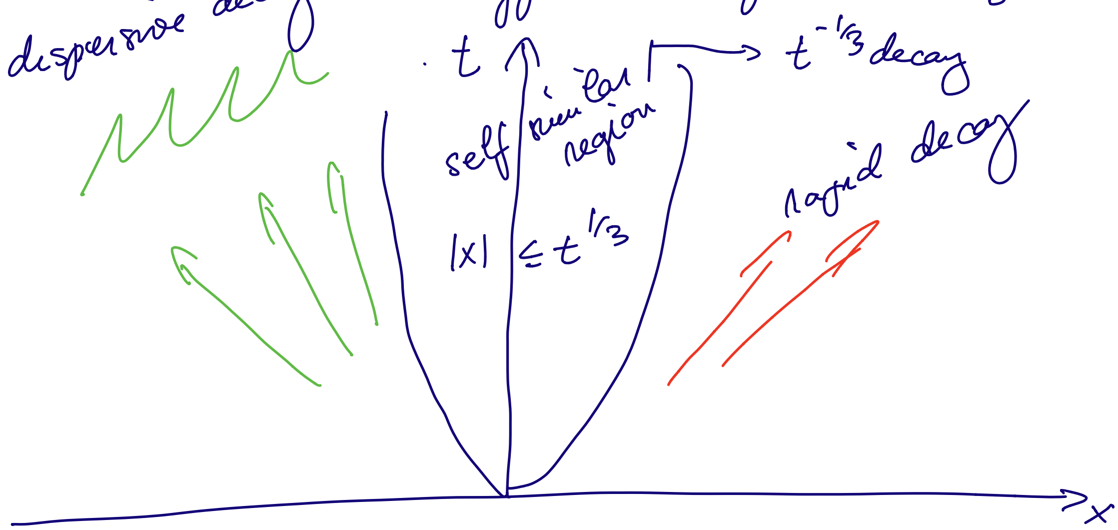
or any other Sobolev norm.

Symbol for L:

$$\ell(x,\xi) = x + 3t\xi^2$$

$\to$ elliptic if x is positive

$\to$ hyperbolic if x is negative



Example 3

$$(i\partial_t + A(D))u = 0$$

$$Lu = (x + t\, a_\xi(D))u$$

$$[L, i\partial_t + A(D)] = 0$$

$$\|u\|_{L^2} + \|Lu\|_{L^2} \overset{?}{\geq} c(t)\|u\|_{L^\infty}$$

Nonlinear problems

Example:

$$(i\partial_t - \Delta)u = u|u|^4 \quad \text{in } 1\text{-d.}$$

$$u_0 \in H^1, \qquad x u_0 \in L^2$$

$$\|u_0\|_{H^1} + \|x u_0\|_{L^2} \leq \varepsilon$$

Theorem. If $\varepsilon \ll 1$ then solutions are global, and decay like $t^{-1/2}$.

Proof. Energy conservation:

$$\|u\|_{L^2} + \|\partial u\|_{L^2} \leq \varepsilon$$

$$Lu = (x - 2it\partial_x)u$$

$$\shortparallel$$

Lu solves the linearized equation.

$$(i\partial_t - \Delta) v \approx v \cdot u^4$$

$$\frac{d}{dt} \|v\|_{L^2}^2 \leq \|v\|_{L^2}^2 \cdot \|u\|_{L^\infty}^4$$

$$\|Lu(t)\|_{L^2} \leq \|Lu(0)\|_{L^2} \cdot e^{\int_0^t \|u\|_{L^\infty}^4 dx}$$

Expect: $\|u\|_{L^\infty} \leq \frac{\varepsilon}{\sqrt{t}}$

If this is true, then we get:

② $$\|Lu(t)\|_{L^2} \leq \varepsilon$$

K-S bound:

③ $$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{t}} \|u\|_{L^2} \cdot \|Lu\|_{L^2} \leq \frac{\varepsilon}{\sqrt{t}}$$

Make bootstrap assumption:

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{t}} \quad \left[\frac{\varepsilon^{1/2}}{\sqrt{t}} \right]$$

This implies ② and then ③.

Final comments:

① Sometimes the decay might not be enough to get global solutions.

- Ifrim-T. Benjamin Ono

$$T_\varepsilon = e^{-\frac{c}{\varepsilon}} \quad \{\text{almost global}\}$$

- Ifrim-Koch-T. KdV equation

$$T_\varepsilon = \varepsilon^{-3} \quad \{\text{quartic time}\}$$

Related to solitons / soliton resolution conjecture.

② Classical KS ineq. $\Rightarrow L$ is a linear operator.

KS = linear

BO + KdV:

$$L \longrightarrow L^{NL}$$

linear nonlinear

$$\|u\|_{L^2} + \|L^{NL} u\|_{L^2} \geq c(V) \|u\|_{L^\infty}$$

Nonlinear K-S inequalities!