Summer School Lecture 8

Daniel Tataru

Wave packets

Semilinear dispersive pde's

- Strichartz estimates
- When are Strichartz est. sharp?

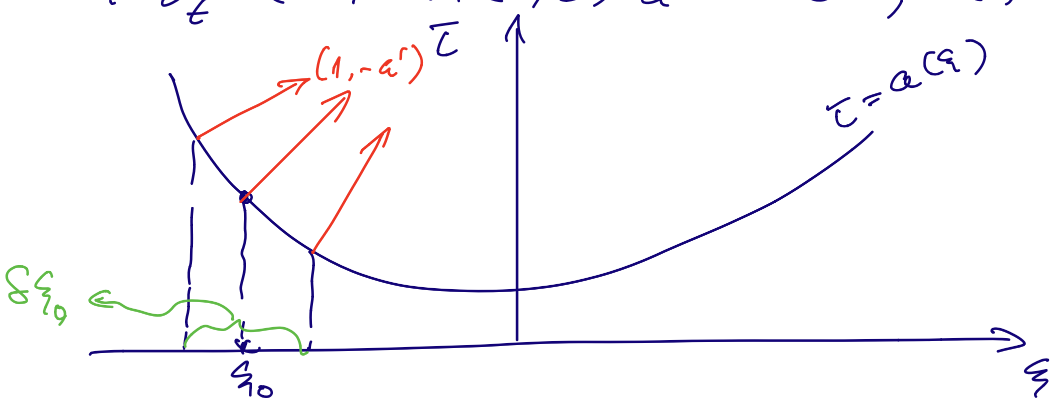
Quasilinear pde's

- Worst nonlinear interactions?

Q: What are the most concentrated solutions for linear dispersive pde's?

Model problem:

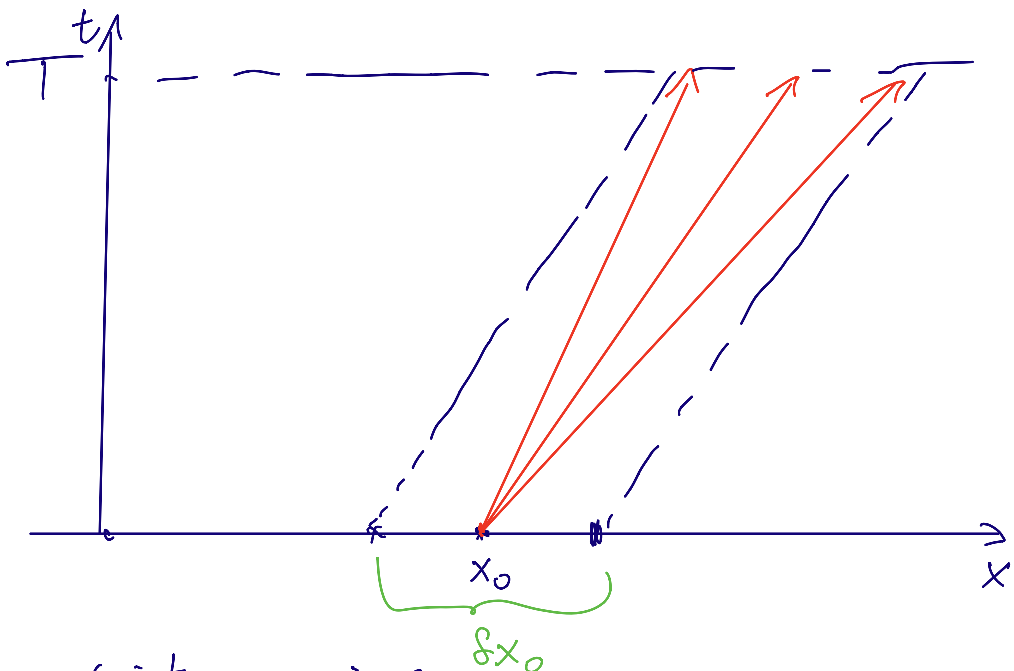
$$i\partial_t u + A(x,D)u = 0 \quad , \quad u(0)=u_0$$



Group velocity:

$$v_{\xi_0} = -a'(\xi_0)$$

Q: Localisation in the physical space



<u>Uncertainty principle</u>:

$$\delta\xi_0 \cdot \delta x_0 \geq 1.$$

= ⇝ strongest concentration

- Find the time scale where dispersion sets in

$$\delta v = |a''(\xi_0)| \cdot \delta\xi_0$$

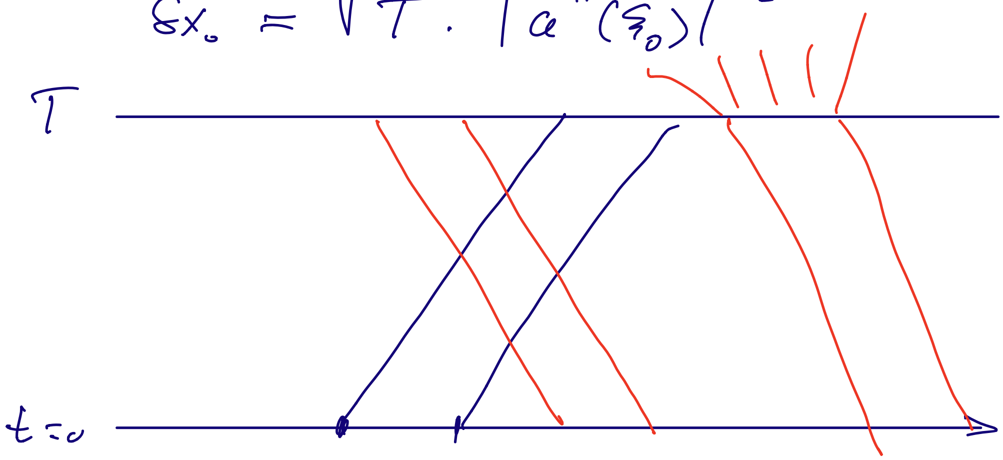
Spread of waves by time T is

$$\delta x = T \cdot \delta v$$

$$\begin{aligned}\delta x_0 &\approx \delta x \\ &= T \cdot \delta v \\ &= T \cdot |a''(\xi_0)| \cdot \delta \xi_0\end{aligned}$$

$$(\delta x_0)^2 \approx T \cdot |a''(\xi_0)|$$

$$\delta x_0 \approx \sqrt{T} \cdot |a''(\xi_0)|^{1/2}$$



$$\text{wave packet} \approx A \; e^{i\phi}$$

amplitude — important in nonlinear problems

phase, solves eikonal eqn. (linear for the simplest wave packets)

- Gaussian beams
- coherent states

Move return to nonlinear pde's

- long time solutions
- global solutions, scattering

$\Downarrow$

as $t \to \infty$, sln looks like a linear wave

- high power nonlin $\longrightarrow$ low power

$\Downarrow$ scattering $\qquad \Downarrow$?

<u>Model problem</u>

$$(i\partial_t - \Delta) u = \pm u \cdot |u|^2 \qquad 1\text{-}d$$

Q: Is there scattering?

$$\underbrace{u}_{\text{linear object}} \cdot \underbrace{|u|^2}_{\text{coeff.}} \longrightarrow \frac{1}{t} \text{ dispersive decay.}$$

$\Downarrow$

not integrable at ∞.

$\to$ no scattering can happen

$\to$ borderline case ($\frac{1}{t}$ almost integrable)

$\to$ NLS approximation problem

Linear scattering:

$$a_\infty(v)$$
$$\uparrow t \to \infty$$
$$u(t,x) \approx \frac{1}{\sqrt{t}} \cdot e^{i\phi(t,x)} \cdot a(t,v)$$

$$\phi(t,x) \approx t \cdot \psi(v) \qquad v = \frac{x}{t}$$

What if we plug into the nonlinear equation?

$$\phi(t,x) = \frac{x^2}{4t} = t \cdot \underbrace{\left(\frac{v^2}{4}\right)}_{=\psi}$$

$\to$ $O\left(\frac{1}{\sqrt{t}}\right)$ cancels from eikonal eqn.

$\to$ Next term $O\left(\frac{1}{t^{3/2}}\right)$

– would cancel for linear eqn.

$\to$ $\frac{1}{t^{3/2}}$ terms in the nonlinear eqn:

$$e^{i\phi} \frac{1}{t^{1/2}} \Big(i \underbrace{a_t(t,v)}_{O(\frac{1}{t})} + \frac{1}{t} \cdot \underbrace{a \cdot |a|^2}_{} \Big)$$

$\downarrow$ ∂_t on a

we want

$a_t \in \ell^1$

$a \to a_\infty$

$\downarrow$ $\frac{1}{t} \underbrace{a_\infty^3}_{}$ $\notin L^1$

Asymptotic equation for a:

$$a_t = \frac{i}{t} a|a|^2 \qquad \text{ode for } a.$$

Can eliminate t if we change variables. to

$$s = \log t$$

$$a_s = i\, a \cdot |a|^2 \longrightarrow |a| \text{ stays constant.}$$

$$a(s) = a(0) \cdot e^{is|a|^2}$$

Modified ansatz at ∞:

$$a \to \gamma$$

$$u(t,x) \simeq \frac{1}{\sqrt{t}}\, e^{i\phi} \cdot a(v) \cdot e^{i|a|^2 \ln t}$$

$\boxed{\text{Modified scattering}}$

Theorem: Assume

$$\|u_0\|_{L^2} + \|x u_0\|_{L^2} \le \varepsilon \ll 1.$$

Then the solution is global and has modified scattering asymptotics.

"Proof" (outline)

Look for solutions like this:

$$u(t,x) \approx \frac{1}{\sqrt{t}}\, e^{i\phi} \cdot \gamma(t,v)$$

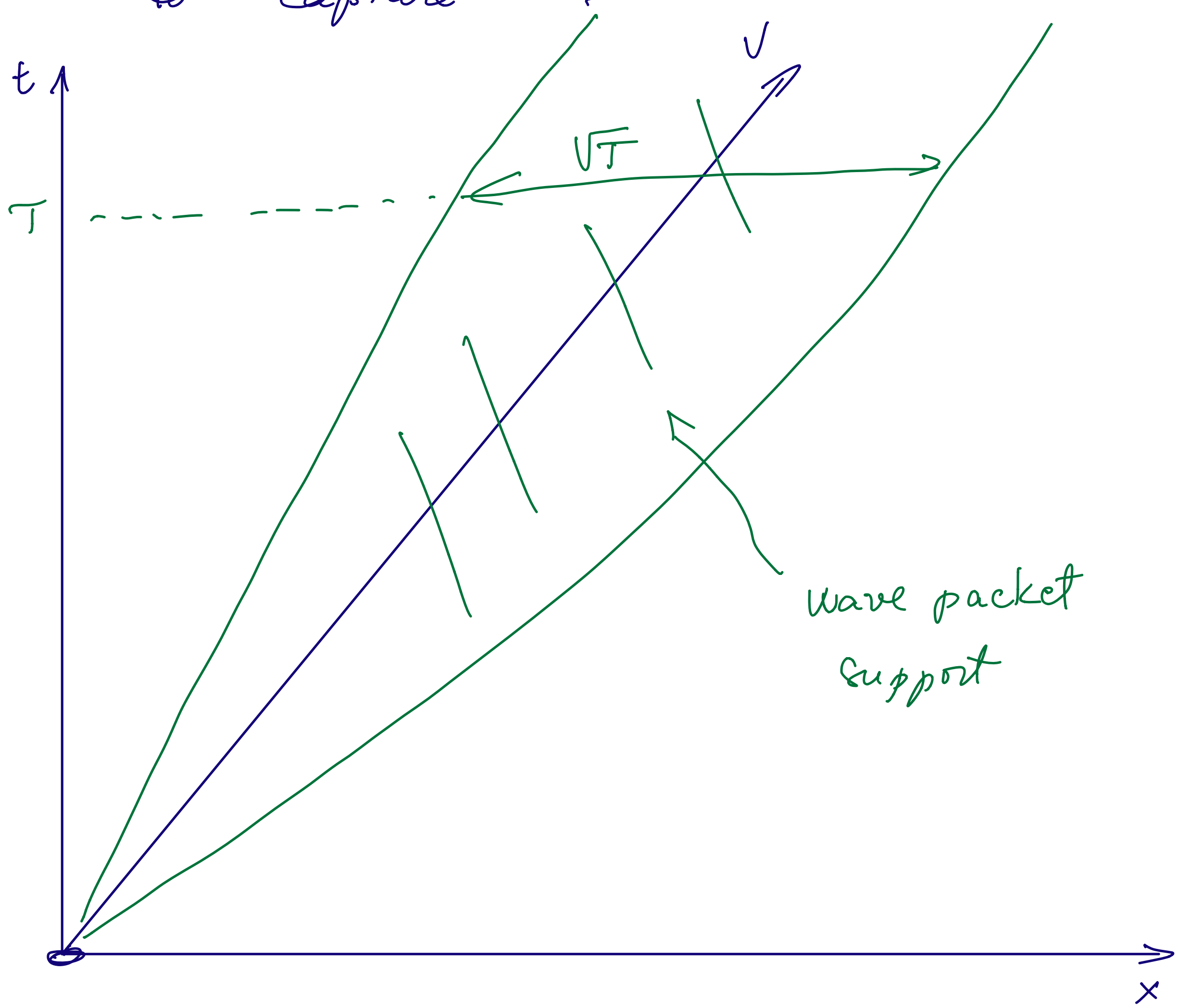
Capture the asymptotic equation for γ:

$$\partial_t \gamma = \frac{1}{t} \cdot \gamma \cdot |\gamma|^2 + \text{err.}$$

$$\uparrow \; L^1_t$$

good variables $= v$, $\ln t$

Key idea: Use wave packets ~~to~~ to capture γ:



Time dependent wave packet scale:

$$\delta x = \sqrt{T} \cdot \sqrt{a''(\xi)}$$
$$\simeq \sqrt{T}$$

Wave packet approximate solution to the linear Schrodinger equation:

$$q_v = e^{i\phi(x,t)} \cdot \chi\left(\frac{x - vt}{\sqrt{t}}\right)$$

not as good almost exact solution to the Schrodinger equation

$$i q_t - \Delta q \approx \frac{1}{t} \cdot q$$
$$\downarrow$$
$$\notin L^1$$

Definition of γ:

$$\gamma(t, v) = \langle q_v, u \rangle$$

$\boxed{\text{Wave packet testing}}$