Summer School Lecture 9

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Recall model problem:

$$i u_t - \Delta u = \pm u |u|^2 \qquad 1-d.$$

$\boxed{\begin{array}{l}\textbf{Theorem:}\ \text{Assume} \\ \qquad \|u_0\|_{L^2} + \|x u_0\|_{L^2} \leq \varepsilon \ll 1. \\ \text{Then the solution is global and} \\ \text{has modified scattering asymptotics:} \\ \qquad u(t,x) \simeq \frac{1}{\sqrt{t}} e^{i\phi} \cdot a(v) \cdot e^{i|a|^2 \ln t}\end{array}}$

$x^{\frac{1}{2}+\delta}$ would be optimal.

Look for solutions like this:

$$u(t,x) \approx \frac{1}{\sqrt{t}} e^{i\phi} \cdot \gamma(t,v) + \underbrace{err}_{\frac{1}{t^{\frac{1}{2}+\varepsilon}}}$$

Q: Given u, how to define γ so that it satisfies the asymptotic system:

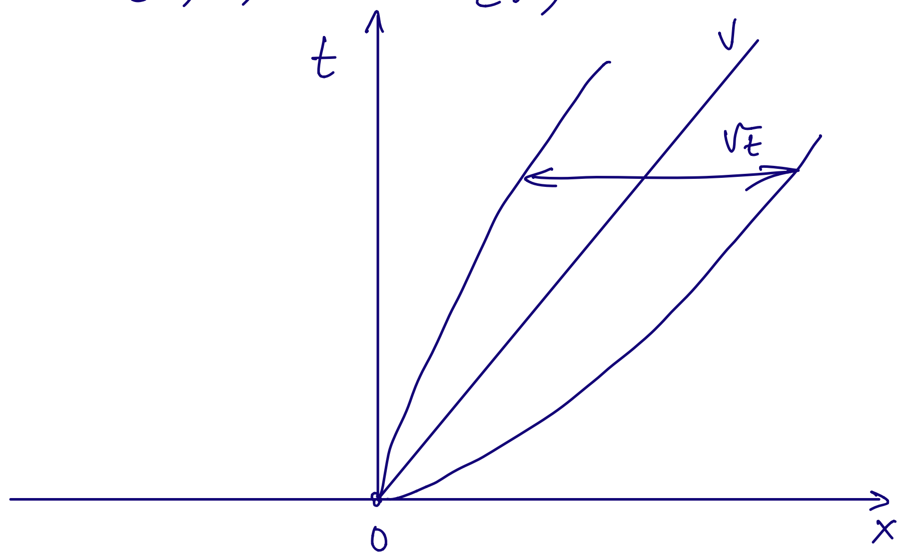
$$\partial_t \gamma = \frac{i}{t} \cdot \gamma \cdot |\gamma|^2 + err.$$

Wave packet testing

$$\varphi_v(x,t) = e^{i\phi(x,t)} \cdot \chi\left(\frac{x - vt}{\sqrt{t}}\right)$$

Definition of γ:

$$\gamma(t,v) = \langle \varphi_v, u \rangle$$



<u>Obs</u>. Suppose u_1, u_2 sol's to linear Schrodinger equation. Then:

$$\frac{d}{dt} \langle u_1, u_2 \rangle = 0.$$

Plug into linear Schrodinger equation

$$i u_t + \Delta u = \pm |u|^2 \cdot u$$

$$i q_t + \Delta q = \text{"} \frac{1}{t} \cdot q \text{"} = err(q)$$

Then

$$(\dot{\gamma} =) \frac{d}{dt} \gamma = \langle q_v, |u|^2 u \rangle + \langle err(q), u \rangle$$

comes from nonlinearity

||

$$\frac{i}{t} \gamma |\gamma|^2$$

+

$\boxed{\text{nonlinear error}}$

$\boxed{\text{linear error}}$

$$\dot{\gamma} = \frac{i}{t} \gamma |\gamma|^2 + \text{nonlin error} + \text{linear error}.$$

History of NLS problem:

- inverse scattering → Deift–Zhou "steepest descent method"
- Hayashi & Naumkin, Kato & Pusateri
  → look at evolution of $\hat{u}(t, \xi)\big|_{\xi = \xi_v} \approx \gamma(t, v)$

small linear errors, but large nonlinear errors.

- Lindblad & Soffer

  Look at $u(t, vt)$

  small nonlinear errors

  large linear errors

Key advantage of wave packet testing:

$$\boxed{\text{balances perfectly linear and nonlinear errors.}}$$

Setup for the proof: $\boxed{\text{bootstrap argument.}}$

We hope for

$$|u(t,x)| \lesssim \frac{\varepsilon}{\sqrt{t}}$$

Set our bootstrap assumption ~~to~~

$$|u(t,x)| \leq \frac{C \cdot \varepsilon}{\sqrt{t}}$$

Use energy est (vector fields) for initial step:

Energy estimates

(a) $$\|u(t)\|_{L^2} = \|u(0)\|_{L^2} \leq \varepsilon.$$

(b) Vector fields:

$$L = x + 2it\partial_x$$

- generator of Galilean group:

If u solves (NLS) then $v = Lu$ solves the linearized equation.

$$iv_t - \Delta v = 2v \cdot |u|^2 + \bar{v} \cdot u^2$$

$$\frac{d}{dt}\|v\|_{L^2}^2 = 2\,\mathrm{Re}\int i\bar{v}^2 u^2\,dx$$

$$\lesssim \underbrace{\|u\|_{L^\infty}^2}_{\frac{c^2\varepsilon^2}{t}} \cdot \|v\|_{L^2}^2$$

$$\|v(t)\|_{L^2} \leq t^{c^2\varepsilon^2}\|v(0)\|_{L^2} \leq \varepsilon\, t^{c^2\varepsilon^2}$$

$$\|u(t)\|_{L^\infty} \leq \frac{1}{\sqrt{t}}\|u(t)\|_{L^2}^{1/2} \cdot \|Lu(t)\|_{L^2}^{1/2}$$

$$\|u(t)\|_{L^\infty} \leq \frac{\varepsilon}{\sqrt{t}} \, t^{c^2\varepsilon^2}$$

Remark: Knowledge of L^2 bound for Lu allows us to approximate u using γ:

$$(*) \qquad u(t,x) = \frac{1}{\sqrt{t}} e^{i\phi} \gamma(t,v) + \underbrace{\text{err.}}_{\text{almost } \frac{1}{t}}$$

$$\sqrt{t}\, u(t,x)\, e^{-i\phi} \xrightarrow{\hspace{2em}} \gamma$$

regularization on the $\sqrt{t}$ scale in x

$\Updownarrow$

$\frac{1}{\sqrt{t}}$ scale in v

$$\sqrt{t}(Lu)e^{-i\phi} \simeq t\partial_x\left(\sqrt{t}\, u\, e^{-i\phi}\right)$$

In a similar manner, we also have:

$$\hat{u}(t,\xi_v) \approx \gamma(t,v) + \underbrace{\text{err.}}_{\text{almost } t^{-1/2}}$$

(*) allows us to bound the nonlinear error in the asymptotic equation.

How about the linear error?

$$\text{Linear error} = \int \underbrace{(i\partial_t - \Delta)\,\varepsilon_v}_{} \cdot \bar{u}\, dx$$

$$\approx \frac{1}{t}\,\varepsilon_v \qquad \left[\varepsilon_v = e^{i\phi} \cdot \chi\left(\frac{x-vt}{\sqrt{t}}\right)\right]$$

$\downarrow$ more accurate computation:

$$\frac{1}{t} \cdot e^{i\phi}\, \underline{\chi'}\left(\frac{x-vt}{\sqrt{t}}\right)$$

exact derivative.

$$\text{Lin. error} = \frac{1}{t}\int \underline{\chi'}\left(\frac{x-vt}{\sqrt{t}}\right) \cdot \overline{e^{-i\phi}\, u}\, dx$$

$$= \frac{1}{t}\int \chi\left(\frac{x-vt}{\sqrt{t}}\right) \cdot \overline{\partial_x\left[e^{-i\phi} u\right]}\, dx$$

$$\left[\phi = \frac{x^2}{4t}\right] \qquad \partial_x\left[e^{-i\phi} u\right] = \frac{1}{t}\, e^{-i\phi} L u$$

Punch line:

$$i\dot{\gamma} \approx \frac{1}{t}\gamma\cdot|\gamma|^2 + \text{err.}$$

$$\text{err.} \Downarrow \frac{\varepsilon}{t^{1+}}$$

$\Downarrow$

ode bound starting at $t=1$.

$\Downarrow$

$$\dot{\gamma} \leq \varepsilon$$

Modified scattering thm:

$$(*)(*) \quad u \simeq \frac{1}{\sqrt{t}} e^{i\phi} a(v) e^{i \log t |a(v)|^2}$$

$$u_0 \xrightarrow[\text{Scattering}]{} a(v)$$

data $\qquad\qquad$ $\Downarrow$ asymptotic profile

Asymptotic completeness

$$a(v) \longrightarrow u \to u_0$$

"Cauchy problem with data at ∞"

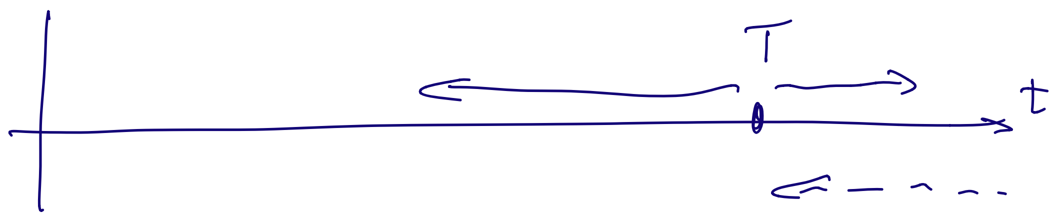
"Proof" Start with a,

Construct u^{app} (*)(*)

$\Downarrow$

close but not quite

Match with an exact sln.



- solve for u_T with data

$$u_T(T) = u^{app}(T)$$

- look for the limit

$$u_\infty = \lim_{T \to \infty} u_T$$