Summer school Lecture 10

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Global solutions for water waves

1. Gravity waves, infinite bottom

(W, Q) (WW) $\begin{cases} W_t + F(1+W_\alpha) = 0 \\ Q_t + FQ_\alpha - igW + P\left[\frac{|Q_\alpha|^2}{J}\right] = 0 \end{cases}$

$\downarrow$

negative frequencies

Differentiated equation

(WW-diff) $\begin{cases} (\partial_t + b\partial_\alpha) W_\alpha + \frac{1+W_\alpha}{1+\bar{W}_\alpha} \cdot R_\alpha + \dots = 0 \\ (\partial_t + b\partial_\alpha) R - i(g+a) W_\alpha + \dots = 0 \end{cases}$

Good variables (W_α, R)

$\mathcal{H}^0 = L^2 \times \dot{H}^{1/2} \Rightarrow$ Energy space for (W, Q)

- also energy space for (W_α, R).
- Higher order spaces $\mathcal{H}^n$
- $\mathcal{H}^1 \ni (W_\alpha, R) \Rightarrow$ local well-posedness

<u>Thm.</u> Suppose that the initial data for (WW) satisfies:

$$\|(W,Q)\|_{H^0} + \|(W_\alpha, R)\|_{\mathcal{H}^5} + \|x(W_\alpha, R)\|_{\mathcal{H}^0} \leq \varepsilon$$

Then the solutions (W,Q) are global in time, and they satisfy pointwise decay bounds:

$$A, B \leq \frac{\varepsilon}{t^{1/2}}.$$

$$A = \|W_\alpha\|_{L^\infty} + \|D^{1/2} R\|_{L^\infty}$$

$$B = \|D^{1/2} W_\alpha\|_{BMO} + \|\partial_\alpha R\|_{BMO}$$

- energy estimates also
- new work w. Mihaela and Albert down to optimal regularity threshold.
- Modified scattering

History of the problem:

- Wu - '09 ⟹ almost global result
  $T_\varepsilon = e^{\frac{c}{\varepsilon}}$
- holom. coords. + Lagrangian coord.

- 3.d ⇒ Wu, Germain - Masmoudi - Shatah
space-time resonances method
- Ionescu - Pusateri (Wu's set-up + Eulerian)
'13
- Alazard + Delort - paradiff. { partial normal form. / paradiagonaliz.
'13
- our work → 6 month later

② Capillary waves in infinite depth

Theorem Small + localized data ⇒
⇒ global result, + pointwise $t^{-1/2}$
decay + modified scattering.

- Ionescu - Pusateri ⇒ alternate proof

Main steps of the proof

- energy estimates → modified energies
- Vector field pointwise decay bounds
- wave packet testing.

Vector fields:

- linear problem
- nonlinear problem
  - adapt VF to nonlinear problem.

Scaling of WW. eqn.

$$\left.\begin{array}{l} W(t,x) \\ Q(t,x) \end{array}\right\} \longrightarrow \begin{array}{l} \lambda^{-1} W(\lambda^{1/2} t, \lambda x) \\ \lambda^{-3/2} Q(\lambda^{1/2} t, \lambda x) \end{array}$$

$$\Downarrow$$

$$(W_\lambda, Q_\lambda)$$

(w, r) good var.

$$\Uparrow$$

$$(w, q) = \frac{d}{d\lambda} (W_\lambda, Q_\lambda)_{\lambda=1} \quad := \underline{\underline{S}}(W, Q)$$

vector field.

$$\Downarrow$$

Solution to the linearized eqn.

nonlinear

Linear part:

$$S\begin{pmatrix} W \\ Q \end{pmatrix} = \begin{pmatrix} \alpha W_\alpha - t Q_\alpha \\ \alpha Q_\alpha - itg W \end{pmatrix} + \text{nonlinear}$$

$\longrightarrow$ similar scaling VF also for capillary wave.

Energy estimates

ⓐ For the full equation:

$$E^{n}(W_\alpha, R) \approx \|(W_\alpha, R)\|^2_{\mathcal{H}^n}$$

$$\frac{d}{dt} E^{n}(W_\alpha, R) \lesssim_A A B \|(W_\alpha, R)\|^2_{\mathcal{H}^n}$$

ⓑ For the linearized equation:

$$E^{lin}(w, r) \simeq \|(w, r)\|^2_{\mathcal{H}^0}$$

$$\frac{d}{dt} E^{lin}(w, r) \lesssim_A A B \|(w, r)\|_{\mathcal{H}^0}$$

As a conclusion we want:

$(*)$ $\qquad A, B \leq \varepsilon t^{-1/2}$

Frame the proof as a bootstrap argument.

Bootstrap assumption:

$$A, B \leq C \cdot \varepsilon t^{-1/2}$$

Remember at the end to insure we have proved $(*)$!

Apply the bootstrap assumption in energy estimates:

$$(EE)\quad \begin{cases} \|(W_\alpha, R)\|_{\mathcal{H}^n} \leq \varepsilon\, t^{C^2\varepsilon^2} \\ \|(w, r)\|_{\mathcal{H}^0} \leq \varepsilon\, t^{C^2\varepsilon^2} \end{cases}$$

Can neglect the growth if

$$t^{C^2\varepsilon^2} \lesssim 1 \quad \Leftarrow \quad t \leq e^{\frac{c}{\varepsilon^2}}$$

(almost global)

Next: pointwise bounds from energy:

$$(EE) \Rightarrow \quad A, B \leq \varepsilon\, t^{C^2\varepsilon^2} \cdot \frac{1}{\sqrt{t}}\, c(v)$$

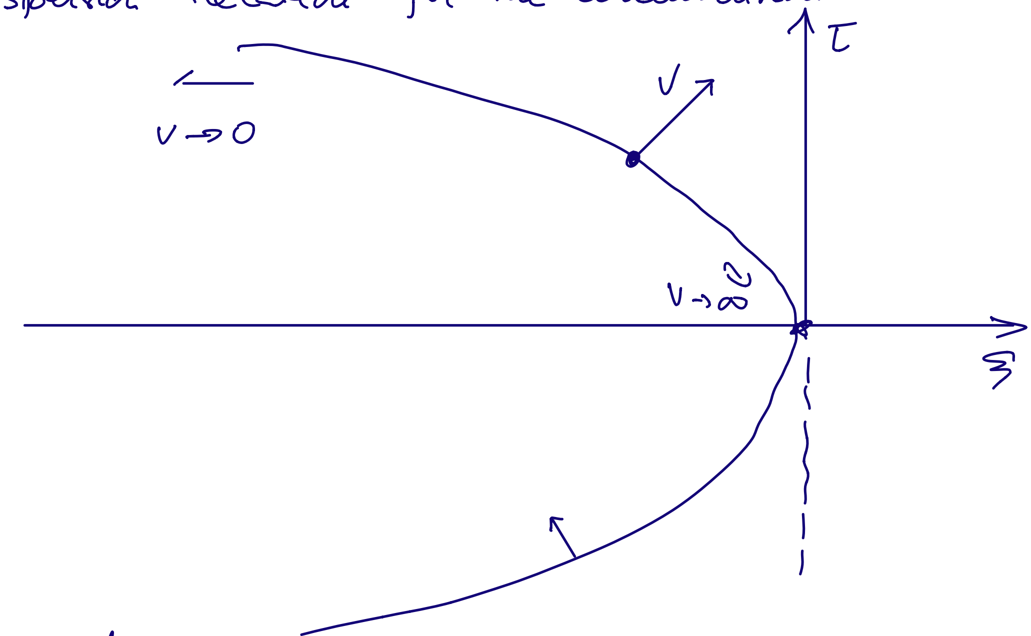
gain as $v \to \infty$

gain as $v \to 0$

- work at level of normal form variables

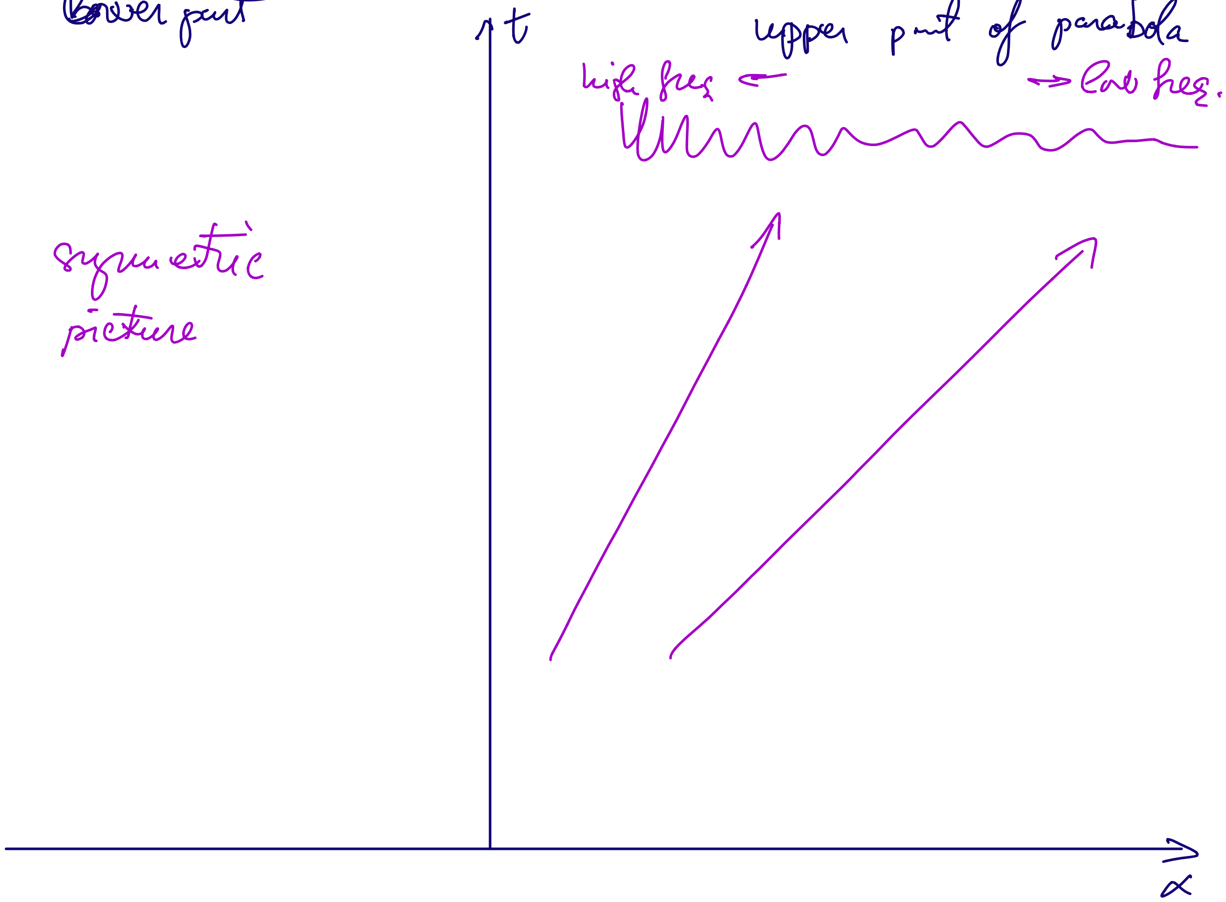
Proof in pictures:

Dispersion relation for the linearization around 0.



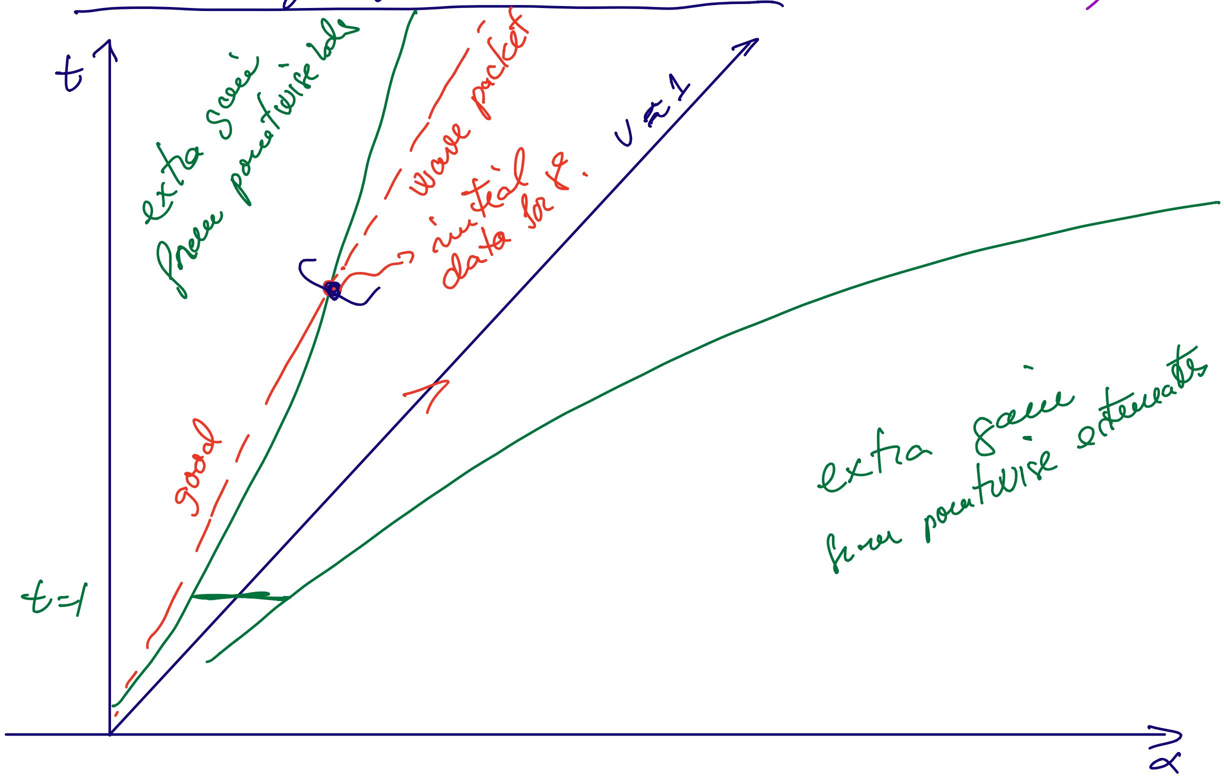
lower part

upper part of parabola



symmetric picture

Picture for pointwise bounds (right half)



Wave packet method,

$$i\gamma_t = \frac{c(v)}{t}\gamma|\gamma|^2 + err.$$

$$\gamma(t,v)$$

Wave packets: $\begin{pmatrix} \mathring{w}_v \\ \mathring{q}_v \end{pmatrix}$

$\rightarrow$ system, but wave packets constructed as in the scalar case.

To whom do we apply $\begin{pmatrix} \mathring{W} \\ \mathring{\xi} \end{pmatrix}$?

→ to the normal form variables $(\tilde{W}, \tilde{Q})$

$$\tilde{W}_t + \tilde{Q}_\alpha = W^{[3]} + W^{[4+]}$$

$$\tilde{Q}_t - ig\tilde{W} = Q^{[3]} + \underbrace{Q^{[4+]}}_{\substack{t^{-1-} \text{ decay} \\ \text{can be neglected}}}$$

Structure of cubic terms:

$$W^{[3]} = L_{hhh}(W, W, W)$$
$$+ L_{hah}(W, \bar{W}, W)$$
$$+ L_{haa}(W, \bar{W}, \bar{W})$$
$$+ \cancel{L_{aaa}(\bar{W}, \bar{W}, \bar{W})}$$

Resonance analysis along a ray $x = v \cdot t$.

Frequency $\xi \approx \xi_v$

$L_{hhh}(W, W, W) \implies$ nonresonant.

$\xi_v \quad \xi_v \quad \xi_v \implies 3\xi_v$

$\sqrt{\xi_v} \quad \sqrt{\xi_v} \quad \sqrt{\xi_v} \implies 3\sqrt{\xi_v}$ } not on parabola

$L_{haa} \Rightarrow$ nonresonant.

$(W, \bar{W}, \bar{W})$

$\xi_v \quad -\xi_v \quad -\xi_v \quad \Rightarrow \quad -\xi_v$

L_{hah} = resonant !

↓

need to compute

↓

contributes to the constant in asymptotic equation.

- solitary waves → obstructions to global solutions

still open in 3D :)

Thm There are no solitary waves for these two problems.

Ifrim - T. 2018

- gravity, deep water
- capillary, deep water

$\left\{\begin{array}{l}\text{Shallow water} \\ \text{Constant vorticity}\end{array}\right. \Bigg\|$ small solitons

- soliton resolution conjecture applies !!

— The End —