Lecture 3

Aug 4, 2020

Last time:

Theorem (CKV) Assume $v_0 \in H^{r+1}$ where $r > \frac{d}{2}$, and assume

$$\nabla v_0 \in G_{s,\delta}$$

for some $\delta > 0$ and Gevrey index $s \geq 1$. Then $\exists\, T > 0$ and a unique sol'n

$$(v, \partial) \in C([0,T], H^{r+1}(\mathbb{R}^d)) \times C([0,T], H^r(\mathbb{R}^d))$$

s.t.

$$(\nabla v, \partial) \in L^\infty([0,T], G_{s,\delta})$$

$$\|f\|_{G_{s,\delta}} = \sum_{m \geq 0} \frac{\delta^m}{m!^s} \|\partial_1^m f\|_{H^r}$$

($s=1$: analytic
$s \geq 1$: Gevrey)

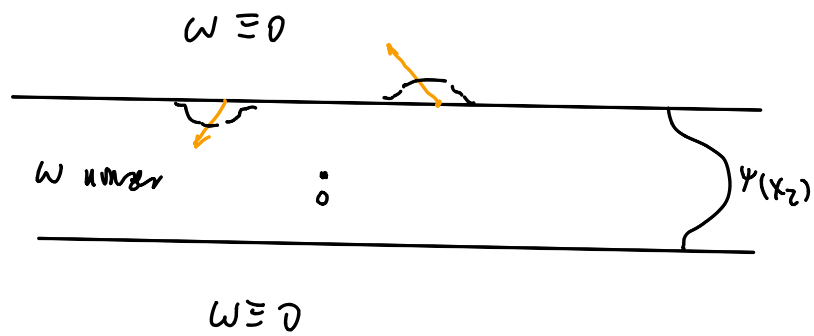
1) Anisotropic analytic regularity fails in the Eulerian framework:

Choose

$$\omega^{(k)}(x_1, x_2) = \varepsilon_0\, k\, e^{-k^2(x_1^2 + x_2^2)}\, \psi(x_2), \quad k = 1, 2, \ldots$$

As $k \to \infty$ this converges to δ_0 as $k \to \infty$

ψ: compactly supported in x_2.



2) Ill-posedness in the analytic space $G_{1,0}$

DiPerna–Majda: For any f, g, the fn

$$u(x_1, x_2, x_3, t) = \left(f(x_2), 0, g(x_1 - t f(x_2))\right)$$

solves the 3D Euler $\forall f, g$. Initial data

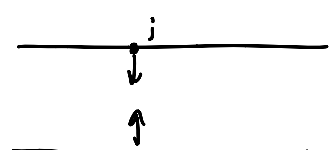
$$u_0(x_1, x_2, x_3) = \left(f(x_2), 0, g(x_1)\right)$$

(Bardos–Titi: ill-posedness of the Euler in $C^{0,\alpha}$)

Take

$$g \approx \frac{1}{1+x^2}$$ complex singularity at i

$$f = \sin x$$



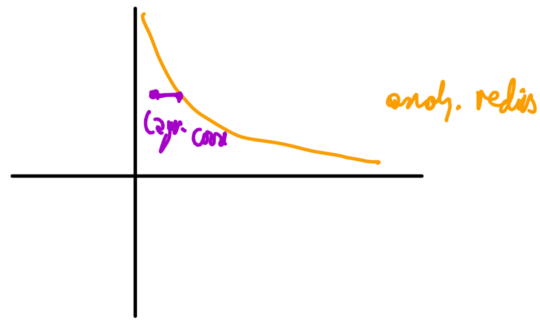
To get the correct g, we integrate four times and remove the low moments & periodize.

Ex (Shnirelman): Cellular flow

$$u(x_1, x_2) = (\sin x_1 \cos x_2, \; -\cos x_1 \sin x_2)$$

stationary sol'n of the EE which is entire.

In Lagrangian coordinates:

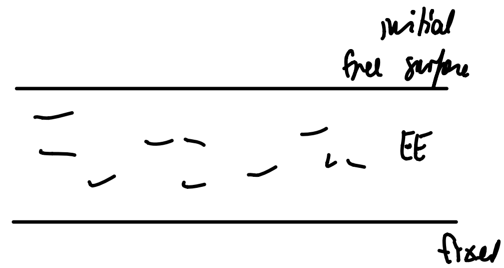


Euler Equations with Free Interface

curl u_0 need not be zero.

Consider the EE in

$$\Omega = \mathbb{T}^2 \times (0,1) \subseteq \mathbb{R}^3$$
$$\Gamma_1 = \mathbb{T} \times \{1\}$$
$$\Gamma_0 = \mathbb{T} \times \{0\}$$



Initially assume $h = 0$ (height) for simplicity.

Christodoulou - Lindblad '00
Coutand - Shkoller '07 (loc existence in H^3)
Shatah - Zeng '08 (loc existence in H^3)
Zhang - Zhang
K - Tuffaha ~ 14, K-Tuffaha - Vicol 17
$u_0 \in H^{2.5+\delta}$, curl $u_0 \in H^3$

Wang et al (to appear) 2015 preprint
$u_0 \in H^{2.5+\delta}$ $(\sigma = 0)$

Disconzi - K - Tuffaha (19) $u_0 \in H^{2.5+\delta}$, $\sigma > 0$,
& a cond. on free surface

(long time behavior: S. Wu, Alazard - Burq - Zuily, Ifrim - Tataru, ...)

let

$$\eta_t(y,t) = u(\eta(y,t),t)$$

$$\eta(y,0) = y$$

and

$$a = (\nabla\eta)^{-1}$$

Also, denote

$$v(y,t) = u(\eta(y,t),t)$$

$$q(y,t) = p(\eta(y,t),t).$$

Then

$$\begin{cases} \partial_t v_i + a_{ki}\,\partial_k q = 0 \\ a_{ki}\,\partial_k v_i = 0 \end{cases}$$

(compare $\partial_t u + u\cdot\nabla u + \nabla p = 0$, $\nabla\cdot u = 0$)

The particle map η satisfies

$$\eta_t = v$$

$$\eta(y,0) = y$$

The cofactor matrix $a = (\nabla\eta)^{-1}$ satisfies

$$a_t = -a\,\nabla v\,a$$

$$a(y,0) = I$$

($a\,\nabla\eta = I$

$a_t\,\nabla\eta = -a\,\nabla\eta_t$

$= -a\,\nabla v$

$\therefore a_t = -a\,\nabla v\,a$)

So the system reads $(y \to x)$

$$\begin{cases} \partial_t v^i + a_{ki}\, \partial_k q = 0 \\ a_{ki}\, \partial_k v_i = 0 \\ a_t = -\, a\, Dv\, a & \text{(ODE)} \\ \eta_t = v & \text{(ODE)} \end{cases}$$

with the initial conditions

$$\begin{cases} v(0) = v_0 \\ a(0) = I & \text{i.c.} \\ \eta(x,0) = x \end{cases}$$

Boundary condition:

$q = 0$ on $\Gamma_1 = \mathbb{T}^2 \times \{1\}$ (no surface tension)

$v \cdot n = 0$ on $\Gamma_0 = \mathbb{T}^2 \times \{0\}$

$H = \{ v \in L^2 : D \cdot v = 0,\ v \cdot n = 0 \text{ in } \Gamma_0)$

<u>Theorem</u> (KTV) Let $\delta > 0$ be arbitrary. Assume

$$v(0) \in H^{2.5+\delta} \cap H$$

$$\operatorname{Curl} v(0) \in H^{2+\delta}$$

and the Rayleigh-Taylor condition

$$\frac{\partial q}{\partial n}(x,0) \le -\frac{1}{C} \quad \text{on } \Gamma_1.$$

Then $\exists$ sol'n (v, q, η, a) s.t.

$v \in L^\infty H^{2.5+\delta} \cap C H^{2+\delta}$ $(2.5+\delta)$

$\eta \in L^\infty H^{3+\delta}$ $(3+\delta)$

$v_t \in L^\infty H^{2+\delta}$ $(2+\delta)$

$q \in L^\infty H^{3+\delta}$ $(3+\delta)$

$q_t \in L^\infty H^{2.5+\delta}$ $(2.5+\delta)$

on $[0,T]$, where T depends on the initial data.

A priori estimates:

1. Pressure estimates
2. Tangential estimate
3. div-curl estimates on η and v

The boundedness of coefficients.

Lemma Assume that

$\|Dv\|_{L^\infty H^{2.5+\delta}} \leq M$ ← $M = (C \times \text{size of the initial data})$
$= 2\|u_0\|_{H^{2.5+\delta}}$

If

$T \leq \frac{1}{CM}$

then $\|D\eta\|_{H^{1.5+\delta}} \lesssim 1$ (doesn't depend on M) & $\|a\|_{H^{1.5+\delta}} \lesssim 1$ (1)

If $T \leq \frac{\varepsilon}{CM}$, then

(26)

unif. control

$\|a - I\|_{H^{1.5+\delta}} \leq \varepsilon$ $\quad (\Rightarrow \|a - I\|_{L^\infty} \leq \varepsilon)$ (2)

and

$\|a^T a - I\|_{H^{1.5+\delta}} \leq \varepsilon$ needed for the pressure (3)

Recall: $\eta_t = v$

$a_t = -a \nabla v\, a$

All the statements are based on these two ODEs.

Proof (1): $\quad (\nabla \eta_t = \nabla v)$

$$\nabla \eta = I + \int_0^t \nabla v \, ds$$

$\Rightarrow$

$$\|\nabla \eta\|_{H^{1.5+\delta}} \lesssim 1 + Mt \lesssim 1 + MT$$

So if $T \leq \frac{1}{CM}$ $\quad \approx (1)_1$

For a: Jacobian $= 1$ $\quad (\det \nabla \eta = 1)$

$$\|a\|_{H^{1.5+\delta}} \lesssim \|\nabla \eta\|^2_{H^{1.5+\delta}} \lesssim 1$$

(2) We have

$$a - I = \int_0^t a_t = -\int_0^t a \nabla v\, a$$

$\therefore$

$$\|a - I\|_{H^{1.5+\delta}} \lesssim \int_0^t 1 \;\|\nabla v\|_{H^{1.5+\delta}}\; 1 \, ds \leq MT$$

$MT \leq \varepsilon \Rightarrow \|a - I\|_{H^{1.5+\delta}} \lesssim \varepsilon$

(3) We have

$$a a^T - I = \int_0^t a_t a^T + \int_0^t a\, a_t^T$$
$$= - \int_0^t a \nabla v\, a\, a^T - \int_0^t a \quad a^T (\nabla v)^T a^T$$

$$\Rightarrow \quad \| a a^T - I \|_{H^{1.5+\delta}} \lesssim \int_0^T \| \nabla v \|_{H^{1.5+\delta}} \leq M T .$$ ▭

Pressure Estimates

Take the div of the EE and obtain the Laplace eq. for q (or p)

$$(\; u_t + u \cdot \nabla u + \nabla p = 0$$
$$\Rightarrow \cancel{\nabla \cdot u_t}^{\,0} + \nabla \cdot (u \cdot \nabla u) + \Delta p = 0$$
$$\therefore \quad \Delta p = - \nabla \cdot (u \cdot \nabla u) = - \partial_j (u_i \partial_i u_j)$$
$$\Delta p = - \partial_j (u_i \partial_i u_j)$$
$$\Delta p = - \partial_{ij} (u_i u_j) \qquad \text{(useful if no bdry)}$$
$$\Delta p = - \partial_j u_i \; \partial_i u_j$$
$$(\; \Delta p = - \nabla (u \nabla u) \; , \; \Delta p = \nabla^2 (u u) \; , \; \Delta p = \nabla u \nabla u \;))$$

Apply $a_{ji}\partial_j$ to

$$\begin{cases} \partial_t v_i + a_{ki}\partial_k q = 0 \\ a_{ki}\partial_k v_i = 0 \end{cases}$$

We get

$$a_{ji}\partial_j(a_{ki}\partial_k q) = -a_{ji}\partial_j\partial_t v_i$$

$$\therefore \quad a_{ji}\partial_j(a_{ki}\partial_k q) = \partial_t a_{ji}\partial_j v_i$$

↑

close to

$$a_{ji}\partial_j(\delta_{ki}\partial_k q) = \Delta q$$

($\partial_t a$ is "good" since we eliminate one time derivative from the term $\|a\| \lesssim 1 + \int_0^t \|\nabla v\|$, $\|a_t\| \lesssim \|\nabla v\|$)

By the Piola identity

$$\partial_j a_{ji} = 0$$

Cofactor matrices: the divergence of every column is 0 (Evans, p. 462).

$$\partial_j(\underbrace{a_{ji}a_{ki}}\partial_k q) = \partial_t a_{ji}\partial_j v_i$$

↑

$(aa^T)_{ji}$ ($\approx \delta_{ij}$ within ε in $H^{1.5+r}$)