Lecture 4

Recall:

$$\begin{cases} \partial_t v_i + a_{ki}\, \partial_k q = 0 \\ a_{ki}\, \partial_k v_i = 0 \\ a_t = - a\, Dv\, a & \text{(ODE)} \\ \eta_t = v & \text{(ODE)} \end{cases}$$

with the initial conditions

$$\begin{cases} v(0) = v_0 \\ a(0) = I \\ \eta(x,0) = x \end{cases} \qquad \text{i.c.}$$

Sought regularity:

$$v \in H^{2.5+\delta}, \quad \eta \in H^{3+\delta}, \quad a \in H^{2+\delta}, \quad q \in H^{3+\delta}, \quad q_t \in H^{2.5+\delta}$$

Lemma

$$\|q\|_{H^{3+\delta}} \leq P + P \int_0^T \|q_t\|_{H^{2+\delta}}\, ds$$

where P is a polynomial $\|v\|_{H^{2.5+\delta}}$, $\|\eta\|_{H^{3+\delta}}$, $\|v_0\|_{H^{2.5+\delta}}$

and

$$\|q_t\|_{H^{2.5+\delta}} \leq P \left(+ P \int_0^T \|q_t\|_{H^{2+\delta}}\, ds \right)$$

where P is a polynomial

$\|v\|_{H^{2.5+\delta}}$, $\|v_t\|_{H^{2+\delta}}$, $\|q\|_{H^{3+\delta}}$, $\|\eta\|_{H^{3+\delta}}$, $\|v_0\|_{H^{2.5+\delta}}$

<u>Proof</u> Rewrite the pressure eq. as

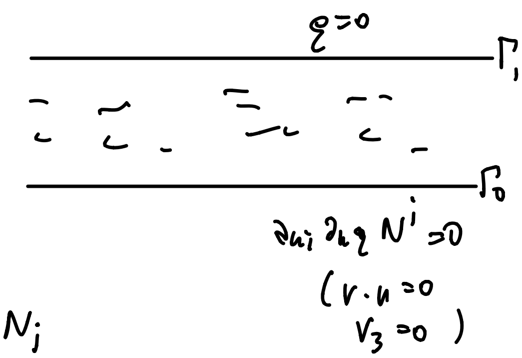
$$\partial_{kk} q = \overset{H^{1+\delta}}{\partial_t a_{ji}}\, \partial_j v_i + \partial_j \Big(\overbrace{(\delta_{jk} - a_{ji}\, a_{ki})}^{(a a)^T_{jk}} \partial_k q \Big)$$

Boundary cond:

$q = 0$ on Γ_1

$a_{ki}\, \partial_k q\, N^i = 0$ on Γ_0

$\hookrightarrow \partial_i q N_i = (\delta_{ki} - a_{ki})\, \partial_k q N_i$



Then apply the elliptic regularity

$$\begin{aligned} \|q\|_{H^{3+\delta}} \lesssim\ & \|\overbrace{\partial_t a_{ji}}^{\nabla v}\, \partial_j v_i \|_{H^{1+\delta}} \\ & + \sum_j \| (\delta_{jk} - a_{ji} a_{ki}) \partial_k q \|_{H^{2+\delta}} \\ & + \| (\delta_{ki} - a_{ki}) \partial_k q\, N^i \|_{H^{1.5+\delta}} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \|q\|_{H^{3+\delta}} & \lesssim \|\nabla v\|_{H^{1+\delta}} \|\nabla v\|_{H^{1.5+\delta}} + \|(I - a a^T) \nabla q\|_{H^{2+\delta}} \\ & \lesssim \quad -//- \quad + \|a\|^2_{H^{2+\delta}} \underbrace{\|\nabla q\|_{H^{1.5+\delta}}}_{\nabla q = \nabla q(0) + \int_0^t \nabla q_t(\tau)\, d\tau} \\ & \qquad + \underbrace{\varepsilon \|\nabla q\|_{H^{2+\delta}}}_{\text{absorb}} \end{aligned}$$

For q_t, differentiate and again apply the estimate $\lesssim \nabla v$

$$\underbrace{\Delta q_t}_{H^{2.5+\delta}} = \partial_{tt} a \nabla v + \partial_t a \nabla v_t + \nabla(\partial_t a \nabla q) + \nabla(\partial_t a \nabla q) + \nabla\left((I - a a^T)\nabla q_t\right) \quad \in H^{0.5+\delta}$$

Recall

$$\partial_t a = - a \nabla v a$$

$$\partial_{tt} a = 2 a \nabla v a \nabla v a - a \nabla v_t a$$

$$\therefore \quad |\partial_{tt} a| \sim |\nabla v|^2, \; |\nabla v_t| .$$

Tangential Estimate

Denote

$$S = \Lambda^{2.5+\delta} \qquad (\Lambda : \text{order 1 tangential derivative})$$

where

$$\Lambda = (I - \Delta_2)^{1/2}$$

$$\Delta_2 = \partial_{11} + \partial_{22} \qquad (\text{2D Laplacian})$$

Lemma For $t \in [0,T]$

$$\approx \| S \eta^3(t) \|^2_{L^2}$$

$$\| S v(t) \|^2_{L^2} + \| a_{3i} S \eta^i(t) \|^2_{L^2(\Gamma_1)}$$

$$\leq Q(\|v_0\|_{H^{2.5+\delta}})$$

$$+ \int_0^T P\left(\|v\|_{H^{2.5+\delta}}, \|v_t\|_{H^{2+\delta}}, \|q\|_{H^{3+\delta}}, \|q_t\|_{H^{2.5+\delta}}, \|\eta\|_{H^{3+\delta}}\right) ds$$

Apply S to

$$\partial_t v_i + a_{ki}\,\partial_k q = 0$$

and test with $S v_i$:

$$\frac{1}{2}\frac{d}{dt}\|S v\|_{L^2}^2 = -\int S(a_{ki}\,\partial_k q)\,S v_i$$

$$= -\int \underbrace{S a_{ki}\,\partial_k q\,S v_i}_{=:I_1} - \int \underbrace{a_{ki}\,\partial_k S q\,S v_i}_{=:I_2}$$

$$-\int \underbrace{\big(S(a_{ki}\,\partial_k q) - S a_{ki}\,\partial_k q - a_{ki}\,\partial_k S q\big)}_{\text{OK by Kato-Ponce}}$$

Start with I_2:

$$I_2 \overset{\text{Piola}}{=} \int a_{ki}\,S q\,\partial_k S v_i$$

OK by Kato-Ponce

$$= -\int S q\,\big(S(a_{ki}\,\partial_k v_i) - a_{ki}\,\partial_k S v_i - S a_{ki}\,\partial_k v_i\big)$$

$$-\int S q\,S a_{ki}\,\partial_k v_i$$

$\frac{1}{2}$ derivative off and put it on

$$\|(I-\Delta)^{1/2}(S q\,\partial_k v_i)\|_{L^2}$$

$$\leq \|(I-\Delta)^{1/2} S q\|_{L^2}\,\|\partial_k v_i\|_{L^\infty}$$

$$+ \text{L.O.T}$$

Recall

$$J_1 = -\int S a_{ki}\, \partial_k q\, S v_i$$

Transform : $S a_{ki}$ using $S D\eta$

From

$$a\, D\eta = I$$

we get

$$S a\, D\eta = -a\, S D\eta - \Big(\underbrace{S(a\, D\eta) - S a\, D\eta - a\, S D\eta}_{\substack{\text{OK} \\ \text{Commutator}}} \Big) \Big| \cdot a$$

$\Rightarrow$

$$S a = -a\, S D\eta\, a + \text{commutator}$$

$\therefore$

$$J_1 = -\int (a\, S D\eta\, a)_{ki}\, \partial_k q\, S v_i + \text{l.o.T.}$$

$$= -\int a_{kl}\, S\, \partial_s \eta_l\, a_{si}\, \partial_k q\, S v_i + \text{l.o.T.}$$

$$= \int \overset{L^\infty}{a_{kl}}\, \overset{L^3}{S \eta_l}\, \overset{L^\infty}{a_{si}}\, \overset{L^6}{\partial_{sk} q}\, \overset{L^2}{S v_i}$$

$$+ \int_{\Gamma_1} a_{kl}\, S \eta_l\, \underset{\uparrow\, J}{a_{si}}\, \partial_k q\, S v_i\, N_s\, d\sigma$$

$$+ 0$$

(on Γ_0: vanishes: $\eta^3 \equiv 0$)

On J: Use $N = (0,0,1)$, $\partial_1 q = \partial_2 q = 0$ (since $q = 0$)

Only consider $k=3$, $s=3$

$$J = \int_{\Gamma_1} a_{3l}\, S\eta_l \,\boxed{a_{3i}}\, \partial_3 q \,\boxed{\underbrace{S v_i}_{= \partial_t S\eta_i}}\, d\sigma$$

$$= \frac{1}{2}\int \frac{d}{dt}\left(a_{3l}\, S\eta_l\, a_{3i}\, S\eta_i \right) \partial_3 q \, d\sigma$$

$-$ TERM where time derivative falls on a (lower Hölder)

$$= \frac{1}{2}\frac{d}{dt}\int a_{3l}\, S\eta_l\, a_{3i}\, S\eta_i \underbrace{\partial_3 q}_{\leq -\frac{1}{C}} \, d\sigma$$

$+$ L.O.T.

$\blacksquare$

To get full regularity use the curl-div

$$\|\nabla f\|_{H^s} \lesssim \|\operatorname{div} f\|_{H^s} + \|\operatorname{curl} f\|_{H^s} + \|f \cdot n\|_{H^{s-3/2}}$$

$s \geq 3/2$

η regularity

Recall the Cauchy invariance

$$\varepsilon_{ijk}\, \partial_j v_m\, \partial_k \eta_m = \omega_{0i} \qquad i = 1, 2, 3$$

Consider

$$\varepsilon_{ijk}\, \underbrace{\partial_j \eta_m}_{\approx \delta_{jm}}\, \partial_k \nabla \eta_m$$

$$= \varepsilon_{ijk}\, \partial_j \eta_m\, \partial_k \nabla \eta_m(0) \swarrow 0$$

$$+ \int_0^t \varepsilon_{ijk}\, \partial_j v_m\, \partial_k \nabla \eta_m \leftarrow \text{use Cauchy invariance}$$

$$+ \int_0^t \varepsilon_{ijk}\, \partial_j \eta_m\, \partial_k \nabla v_m$$

$$= - \int_0^t \varepsilon_{ijk}\, \nabla \partial_j v_m\, \partial_k \eta_m$$

$$+ \nabla \omega_{0i}\, t$$

$$+ \int_0^t \varepsilon_{ijk}\, \partial_j \eta_m\, \partial_k \nabla v_m$$

$$= \nabla w_{0i}\, t + 2 \int_0^t \varepsilon_{ijk}\, \partial_j \eta_m\, \partial_k \nabla v_m$$

$\Rightarrow$

$$(\nabla \operatorname{curl} \eta)_i = \varepsilon_{ijk} (\delta_{km} - \partial_k \eta_m)\, \partial_j \nabla \eta_m$$
$$+ t\, \nabla w_{0i}$$
$$+ \int_0^t \varepsilon_{ijk}\, \partial_k v_m\, \partial_j \nabla \eta_m .$$

Apply the $H^{1+\delta}$ norms on both sides.

We clearly need $t\, \nabla w_{0i}$.

$$\|\operatorname{curl} \eta\|_{H^{2+\delta}} \lesssim t\, \|\nabla w_{0i}\|_{H^{1+\delta}} + B$$
$$+ \int_0^t P(\|\eta\|_{H^{3+\delta}}, \|v\|_{H^{2.5+\delta}}).$$

Treatment of divergence: Use $a_{ki}\, \partial_k v_i = 0$ and $\eta_t = v$.

$$\operatorname{div} \eta = (\delta_{ki} - a_{ki})\, \partial_k \eta_i + \overset{\operatorname{div} x = 3}{3} + \int_0^t \partial_t (a_{ki}\, \partial_k \eta_i)\, ds$$
$$= (\delta_{ki} - a_{ki})\, \partial_k \eta_i + 3 + \int_0^t \partial_t a_{ki}\, \partial_k \eta_i\, ds$$
$$+ \int_0^t \underbrace{a_{ki}\, \partial_k v_i}_{=0}\, ds$$

Apply $H^{2+\delta}$:

$$\|\operatorname{div} \eta\|_{H^{2+\delta}} \lesssim \|I - a\|_{H^{2+\delta}} \|D\eta\|_{L^\infty} + 1$$

$$+ \int^t P\left(\|\eta\|_{H^{3+\delta}}, \|v\|_{H^{2.5+\delta}}\right)$$

To get full reg. of η, use div - curl elliptic estimate.

Treat: v curl v Cauchy invariance

div v div - curl.

boundary regularity

$$\|v_3\|_{H^{2+\delta}(\Gamma_1)} \lesssim \underset{\text{L.O.T.}}{\|v_3\|_{L^2(\Gamma_1)}} + \|\bar\partial_2 v_3\|_{H^{1+\delta}(\Gamma_1)}$$

apply trace

$\rightsquigarrow$ $\boxed{D v_3 \text{ is almost tangential}}$

$\partial_1 v_3$, $\partial_2 v_3$ are tangential

$$\partial_3 v_3 = -\partial_1 v_1 - \partial_2 v_2 + \underbrace{\operatorname{div} v}$$

$\|$

Use $a_{ki} \partial_k v_i = 0$

with surface tension: Disconzi, K, Tuffaha 19

$\|v_0\|_{H^{2.5+\delta}} < \infty$

$\|v_{03}|_{\Gamma_1}\|_{H^{2.5}(\Gamma_1)}$ (ensures $v_{tt} \in C^2$)