Lecture A.9

Linear-Size IOP for Machines

Summer Graduate School on
Foundations and Frontiers of Probabilistic Proofs
2021.08.05

Linear-Size IOPs with Sublinear-Time Verification

We have proved that arithmetic "circuit-like" computations have linear-size IOPs:

for every field $\mathbb{F}$ of size $\Omega(n)$ that is smooth [smoothness is for the LDT]

$$\mathrm{R1CS}(\mathbb{F}) \subseteq \mathrm{IOP}\left[\begin{array}{l} \varepsilon_c = 0,\ \varepsilon_s = 0.5,\ \Sigma = \mathbb{F},\ pt = O(n\log n),\ vt = O(n) \\ k = O(\log n),\ r = O(\log n),\ \ell = O(n),\ q = O(\log n) \end{array}\right]$$

The running time of the verifier is optimal, because just reading the statement takes $\Omega(n)$ time.

Similarly to before if we seek sublinear-time verification we need to consider problems whose description is smaller than computation size.

The holy grail would be a statement like the following:

$$\mathrm{NTIME}(T) \subseteq \mathrm{IOP}\left[\begin{array}{llll} \varepsilon_c = 0 & \Sigma = \{0,1\} & pt = O(T) & vt = \mathrm{poly}(n, \log T) \\ \varepsilon_s = 0.5, & k = *, & \ell = O(T), & q = \mathrm{poly}(\log T) \end{array}\right]$$

This remains a challenging open question.

Instead, we will prove a "large alphabet" relaxation of the theorem:

implies prior theorem with $\ell = O(T \log T)$

<u>theorem:</u> for every field $\mathbb{F}$ of size $\Omega(T)$ that is smooth [smoothness is for the LDT]

$$\mathrm{NTIME}(T, \mathbb{F}) \subseteq \mathrm{IOP}\left[\begin{array}{llll} \varepsilon_c = 0 & \Sigma = \mathbb{F} & pt = O(T\log T) & vt = \mathrm{poly}(n, \log T) \\ \varepsilon_s = 0.5, & k = *, & \ell = O(T), & q = \mathrm{poly}(\log T) \end{array}\right]$$

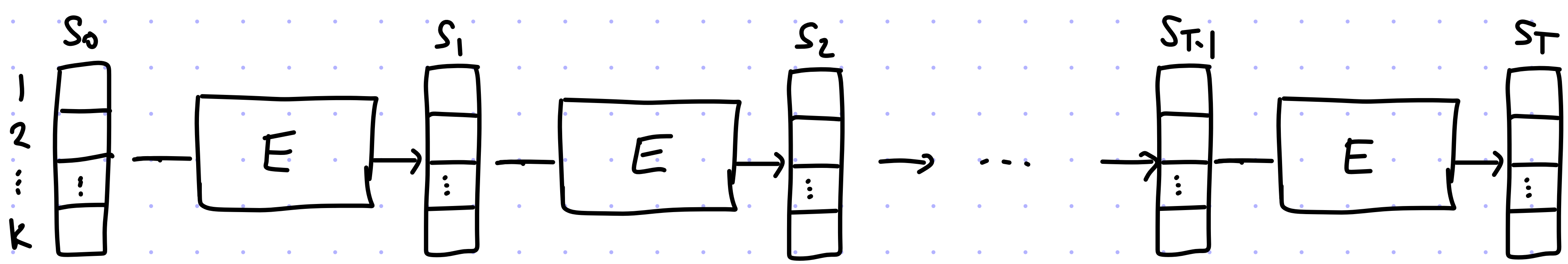
Machine Computations

Informally, a machine is an automaton that can read/write to some type of memory.
If memory = tapes then you get Turing machines.
If memory = RAM then you get register machines (very close to how we think of a computer).

We are going to define languages that model machines that compute over finite fields.
Let's start simple by first doing this for automata (i.e., no memory beyond internal state).

Consider:
- $k \in \mathbb{N}$ — number of internal registers, i.e., a state is $s \in \mathbb{F}^k$
- $E : \mathbb{F}^k \to \mathbb{F}^k$ — transition function mapping current state to next state

A T-step computation:



Specifying the computation requires $O(|E| + \log T)$ bits.
The specified computation involves $O(|E| \cdot T)$ operations, exponentially more in T.
We are in fact interested in non-deterministic computations, and need an appropriate language.

linear proof length means $O(|E| \cdot T)$ ops

Algebraic Automata

The bounded-halting problem for automata:

transition checker — computation input — time bound

<u>def</u>: BH is the set of instances (E, z, T) where $E: \mathbb{F}^{2k} \to \mathbb{F}$, $z \in \mathbb{F}^n$, $T \in \mathbb{N}$ for which $\exists$ execution trace $A_1, \ldots, A_k : [T] \to \mathbb{F}$ s.t.

① the transition checker validates each step: $\forall t \in \{0,1,\ldots,T-1\}$ $E(A_1(t), \ldots, A_k(t), A_1(t+1), \ldots, A_k(t+1)) = 0$

② the first n values of A_1 are z: $A_1|_{[n]} = z$

③ the last value of A_1 is 0: $A_1(T) = 0$

We massage this into a more convenient problem:

- Identify $[T]$ with a multiplicative subgroup $H = \langle \omega \rangle \leq \mathbb{F}$ s.t. $|H| = T$.
  Representing H requires $O(\log|\mathbb{F}|)$ bits rather than $O(|H| \cdot \log|\mathbb{F}|)$.
- Translate the circuit $E: \mathbb{F}^{2k} \to \mathbb{F}$ into quadratic equations $p_1, \ldots, p_m \in \mathbb{F}[X_1, \ldots, X_{2k+\ell}]$ that capture its satisfiability, with $m := O(|E|)$ and $\ell := O(|E|)$ auxiliary variables.

$(E, z, T) \in BH$ iff $\exists$ augmented execution trace $A_1, \ldots, A_k, B_1, \ldots, B_\ell : H \to \mathbb{F}$ — size $(k+\ell)T = O(|E| \cdot T)$

- $\forall t \in \{0,1,\ldots,T-1\}$: $\{p_j(A_1(\omega^t), \ldots, A_k(\omega^t), A_1(\omega \cdot \omega^t), \ldots, A_k(\omega \cdot \omega^t), B_1(\omega^t), \ldots, B_\ell(\omega^t)) = 0\}_{\forall j \in [m]}$
- $A_1|_{H_{in}} = z$, $A_1(\omega^{T-1}) = 0$

Target-on-Subdomain Testing

if $d<|H|$ then $\hat{f}\equiv\hat{z}$, and the problem is an identity test

Consider the setting where the verifier has oracle access to $f: L\to\mathbb{F}$ of degree at most d (with $d\geq|H|$) and wishes to check that $\hat{f}|_H \equiv z$ for a given "target" $z: H\to\mathbb{F}$. (E.g. z is all 0's.)

We have seen this before: $\hat{f}(x)$ agrees with z on H iff $\exists\, \hat{h}(x)$ s.t. $\hat{f}(x)-\hat{z}(x)\equiv\hat{h}(x)v_H(x)$

Hence:

$P((\mathbb{F},L,H,z),f)$

Compute $\hat{h}(x) := \dfrac{\hat{f}(x)-\hat{z}(x)}{v_H(x)}$

$f: L\to\mathbb{F}$

$\xrightarrow{h: L\to\mathbb{F}}$

$V((\mathbb{F},L,H,z))$

- Test that h is δ-close to $RS[\mathbb{F},L,d-|H|]$
- Sample $\gamma\in L$ and check $f(\gamma)-\hat{z}(\gamma)\stackrel{?}{=}h(\gamma)v_H(\gamma)$

Completeness: if $\hat{f}|_H\equiv z$ then $h:=\hat{h}|_L \in RS[\mathbb{F},L,d-|H|]$ and passes check $\forall\gamma\in L$

Soundness: if $\hat{f}|_H\not\equiv z$ then $\forall h: L\to\mathbb{F}$ we have two cases:

- h is δ-far from $RS[\mathbb{F},L,d-|H|] \to$ verifier accepts w.p. $\leq \varepsilon_{LDT}(\delta)$
- h is δ-close to $\hat{h}$ of degree $d-|H| \to \hat{f}(x)-\hat{z}(x)\not\equiv\hat{h}(x)v_H(x)$ so verifier accepts w.p. $\dfrac{d}{|L|}+\delta$

becomes 2δ if f is δ-close to $\hat{f}$

Time complexity of the verifier: [ignore LDT because, if using FRI, time(LDT) $=O(\log|L|)$, which is small]

- if $z\neq 0^H$ then: evaluate v_H at γ and evaluate $\hat{z}$ at $\gamma \to \mathrm{poly}(|H|)$
- if $z=0^H$ then: evaluate v_H at $\gamma \to \mathrm{poly}(|H|)$ in general but $\mathrm{poly}(\log|H|)$ if H is a subgroup!

  E.g. if H is a multiplicative subgroup then $v_H(x)=x^{|H|}-1$. Crucial for us today.

IOP for Algebraic Automata

[L is an evaluation domain disjoint from H]

$P((E,z,T),A)$

- For each $i \in [k]$:

  compute $f_i := \hat{A}_i|_L \in RS[\mathbb{F}, L, |H|-1]$
- Derive auxiliary trace $B_1,\dots,B_\ell : H \to \mathbb{F}$ from the execution trace $A_1,\dots,A_k : H \to \mathbb{F}$
- For each $i \in [\ell]$:

  compute $g_i := \hat{B}_i|_L \in RS[\mathbb{F}, L, |H|-1]$
- For each $j \in [m]$:

  compute $h_j := \hat{h}_j(x)|_L \in RS[\mathbb{F}, L, |H|-1]$

$$\hat{h}_j(x) := \frac{P_j\left(\begin{matrix}\hat{A}_1(x),\dots,\hat{A}_k(x)\\ \hat{A}_1(\omega x),\dots,\hat{A}_k(\omega\cdot x)\end{matrix}, \hat{B}_1(x),\dots,\hat{B}_\ell(x)\right)}{V_H(x)/(x-\omega^{T-1})}$$

- $h_z := \hat{h}_z(x)|_L \in RS[\mathbb{F}, L, |H|-1-n]$

$$\hat{h}_z(x) := \frac{\hat{A}_1(x) - \hat{z}(x)}{V_{H_{in}}(x)}$$

- $h_o := \hat{h}_o(x)|_L \in RS[\mathbb{F}, L, |H|-1-1]$

$$\hat{h}_o(x) := \frac{\hat{A}_1(x) - 0}{x - \omega^{T-1}}$$

$\{f_i : L \to \mathbb{F}\}_{i\in[k]}$

$\{g_i : L \to \mathbb{F}\}_{i\in[\ell]}$

$\{h_j : L \to \mathbb{F}\}_{j\in[m]}$

$h_z : L \to \mathbb{F}$

$h_o : L \to \mathbb{F}$

$\longrightarrow$

$V((E,z,T))$

- Test each received function for the appropriate degree

  [we will come back to this]
- Sample $\gamma \in L$ and check that

– $\forall j \in [m]$

$$h_j(\gamma)\,\frac{V_H(\gamma)}{\gamma - \omega^{T-1}} \stackrel{?}{=} P_j\left(\begin{matrix}f_1(\gamma),\dots,f_k(\gamma)\\ f_1(\omega\cdot\gamma),\dots,f_k(\omega\cdot\gamma)\end{matrix}, g_1(\gamma),\dots,g_\ell(\gamma)\right)$$

– $h_z(\gamma)\,V_{H_{in}}(\gamma) \stackrel{?}{=} f_1(\gamma) - \hat{z}(\gamma)$

– $h_o(\gamma)(\gamma - \omega^{T-1}) \stackrel{?}{=} f_1(\gamma) - 0$

Completeness

Suppose that $A_1, \dots, A_k : H \to \mathbb{F}$ is a witness for $(E, z, T) \in BH$.

- For each $j \in [m]$:

$\forall t \in \{0, 1, \dots, T-1\}$

$$p_j\left(\begin{matrix}\hat{A}_1(\omega^t), \dots, \hat{A}_k(\omega^t) \\ \hat{A}_1(\omega^{t+1}), \dots, \hat{A}_k(\omega^{t+1})\end{matrix}, \hat{B}_1(\omega^t), \dots, \hat{B}_\ell(\omega^t)\right) = 0$$

$\to$ $V_H(X) / (X - \omega^{T-1})$ divides

$$p_j\left(\begin{matrix}\hat{A}_1(X), \dots, \hat{A}_k(X) \\ \hat{A}_1(\omega \cdot X), \dots, \hat{A}_1(\omega \cdot X)\end{matrix}, \hat{B}_1(X), \dots, \hat{B}_\ell(X)\right)$$

$\to$ $\hat{h}_j(X)$ is defined

- $A_1|_{H_{in}} \equiv z \to V_{H_{in}}(X)$ divides $\hat{A}_1(X) - \hat{z}(X) \to \hat{h}_z(X)$ is defined
- $A_1(\omega^{T-1}) = 0 \to X - \omega^{T-1}$ divides $\hat{A}_1(X) - 0 \to \hat{h}_0(X)$ is defined

$P((E, z, T), A)$

- For each $i \in [k]$:
  compute $f_i := \hat{A}_i|_L \in RS[\mathbb{F}, L, |H|-1]$
- Derive auxiliary trace $B_1, \dots, B_\ell : H \to \mathbb{F}$ from the execution trace $A_1, \dots, A_k : H \to \mathbb{F}$
- For each $i \in [\ell]$:
  compute $g_i := \hat{B}_i|_L \in RS[\mathbb{F}, L, |H|-1]$
- For each $j \in [m]$:
  compute $h_j := \hat{h}_j(X)|_L \in RS[\mathbb{F}, L, |H|-1]$

$$\hat{h}_j(X) := \frac{p_j\left(\begin{matrix}\hat{A}_1(X), \dots, \hat{A}_k(X) \\ \hat{A}_1(\omega X), \dots, \hat{A}_k(\omega \cdot X)\end{matrix}, \hat{B}_1(X), \dots, \hat{B}_\ell(X)\right)}{V_H(X)/(X - \omega^{T-1})}$$

- $h_z := \hat{h}_z(X)|_L \in RS[\mathbb{F}, L, |H|-1-n]$

$$\hat{h}_z(X) := \frac{\hat{A}_1(X) - \hat{z}(X)}{V_{H_{in}}(X)}$$

- $h_0 := \hat{h}_0(X)|_L \in RS[\mathbb{F}, L, |H|-1-1]$

$$\hat{h}_0(X) := \frac{\hat{A}_1(X) - 0}{X - \omega^{T-1}}$$

$\{f_i : L \to \mathbb{F}\}_{i \in [k]}$

$\{g_i : L \to \mathbb{F}\}_{i \in [\ell]}$

$\{h_j : L \to \mathbb{F}\}_{j \in [m]}$

$h_z : L \to \mathbb{F}$

$h_0 : L \to \mathbb{F}$

$\longrightarrow$

$V((E, z, T))$

- Test each received function for the appropriate degree
  [we will come back to this]
- Sample $\gamma \in L$ and check that
  - $\forall j \in [m]$
    $h_j(\gamma) \frac{V_H(\gamma)}{\gamma - \omega^{T-1}} \stackrel{?}{=} p_j\left(\begin{matrix}f_1(\gamma), \dots, f_k(\gamma) \\ f_1(\omega \cdot \gamma), \dots, f_k(\omega \cdot \gamma)\end{matrix}, g_1(\gamma), \dots, g_\ell(\gamma)\right)$
  - $h_z(\gamma) V_{H_{in}}(\gamma) \stackrel{?}{=} f_1(\gamma) - \hat{z}(\gamma)$
  - $h_0(\gamma)(\gamma - \omega^{T-1}) \stackrel{?}{=} f_1(\gamma) - 0$

Moreover:

- proof length (in elts): $O((k+\ell+m)|L|) = O((k+\ell+m)|H|) = O(|E| \cdot T)$
- query complexity: $O((k+\ell+m) \cdot \log|L|) = O(|E| \cdot \log T)$
- prover time (in fops): $O((k+\ell+m) \cdot |L| \log|L|) = O(|E| \cdot T \log T)$
- verifier time (in fops): $O((k+\ell+m) \log|L|) + \mathrm{poly}(n) = O(|E| \log T) + \mathrm{poly}(n)$

Soundness

Suppose that $(E, z, T) \notin BH$.

There are two cases:

① One of the functions is far from RS.

- $\exists i \in [k]$ f_i is δ-far from $RS[\mathbb{F}, L, |H|-1]$

or

- $\exists i \in [\ell]$ g_i is δ-far from $RS[\mathbb{F}, L, |H|-1]$

or

- $\exists j \in [m]$ h_j is δ-far from $RS[\mathbb{F}, L, |H|-1]$

or

- h_z is δ-far from $RS[\mathbb{F}, L, |H|-1-n]$

or

- h_o is δ-far from $RS[\mathbb{F}, L, |H|-1-1]$

$\Rightarrow$ verifier accepts w.p. $\le \varepsilon_{LDT}(\delta)$

$P((E, z, T), A)$

- For each $i \in [k]$:

  compute $f_i := \hat{A}_i|_L \in RS[\mathbb{F}, L, |H|-1]$
- Derive auxiliary trace $B_1, \dots, B_\ell : H \to \mathbb{F}$ from the execution trace $A_1, \dots, A_k : H \to \mathbb{F}$
- For each $i \in [\ell]$:

  compute $g_i := \hat{B}_i|_L \in RS[\mathbb{F}, L, |H|-1]$
- For each $j \in [m]$:

  compute $h_j := \hat{h}_j(x)|_L \in RS[\mathbb{F}, L, |H|-1]$

  $$\hat{h}_j(x) := \frac{p_j\left(\begin{matrix}\hat{A}_1(x), \dots, \hat{A}_k(x) \\ \hat{A}_1(\omega x), \dots, \hat{A}_k(\omega \cdot x)\end{matrix}, \hat{B}_1(x), \dots, \hat{B}_\ell(x)\right)}{V_H(x)/(X - \omega^{T-1})}$$
- $h_z := \hat{h}_z(x)|_L \in RS[\mathbb{F}, L, |H|-1-n]$

  $$\hat{h}_z(x) := \frac{\hat{A}_1(x) - \hat{z}(x)}{V_{H_{in}}(X)}$$
- $h_o := \hat{h}_o(x)|_L \in RS[\mathbb{F}, L, |H|-1-1]$

  $$\hat{h}_o(x) := \frac{\hat{A}_1(x) - 0}{x - \omega^{T-1}}$$

$\{f_i : L \to \mathbb{F}\}_{i \in [k]}$

$\{g_i : L \to \mathbb{F}\}_{i \in [\ell]}$

$\{h_j : L \to \mathbb{F}\}_{j \in [m]}$

$h_z : L \to \mathbb{F}$

$h_o : L \to \mathbb{F}$

$\longrightarrow$

$V((E, z, T))$

- Test each received function for the appropriate degree

  [we will come back to this]
- Sample $\gamma \in L$ and check that
- $\forall j \in [m]$

  $$h_j(\gamma) \frac{V_H(\gamma)}{\gamma - \omega^{T-1}} \stackrel{?}{=} p_j\left(\begin{matrix}f_1(\gamma), \dots, f_k(\gamma) \\ f_1(\omega \cdot \gamma), \dots, f_k(\omega \cdot \gamma)\end{matrix}, g_1(\gamma), \dots, g_\ell(\gamma)\right)$$
- $h_z(\gamma) V_{H_{in}}(\gamma) \stackrel{?}{=} f_1(\gamma) - \hat{z}(\gamma)$
- $h_o(\gamma)(\gamma - \omega^{T-1}) \stackrel{?}{=} f_1(\gamma) - 0$

② all functions are close to (unique) polynomials $\{\hat{f}_i\}_{i \in [k]}, \{\hat{g}_i\}_{i \in [\ell]}, \{\hat{h}_j\}_{j \in [m]}, \hat{h}_z, \hat{h}_o$ of the appropriate degree.

ⓘ $\exists j \in [m]$ $\hat{h}_j(x) \frac{V_H(x)}{X - \omega^{T-1}} \not\equiv p_j\left(\begin{matrix}\hat{f}_1(x), \dots, \hat{f}_k(x) \\ \hat{f}_1(\omega \cdot x), \dots, \hat{f}_k(\omega \cdot x)\end{matrix}, \hat{g}_1(x), \dots, \hat{g}_\ell(x)\right) \rightarrow$ consistency test passes w.p. $\le \frac{2|H|-2}{|L|} + (2k+\ell)\delta$

or

ⓘⓘ $\hat{h}_z(x) V_{H_{in}}(x) \not\equiv \hat{f}(x) - \hat{z}(x) \rightarrow$ consistency check accepts w.p. $\le \frac{|H|-1}{|L|} + 2\delta$

or

ⓘⓘⓘ $\hat{h}_o(x)(X - \omega^{T-1}) \not\equiv \hat{f}_1(x) - 0 \rightarrow$ consistency check accepts w.p. $\le \frac{|H|-1}{|L|} + 2\delta$

several options to make this < 1:

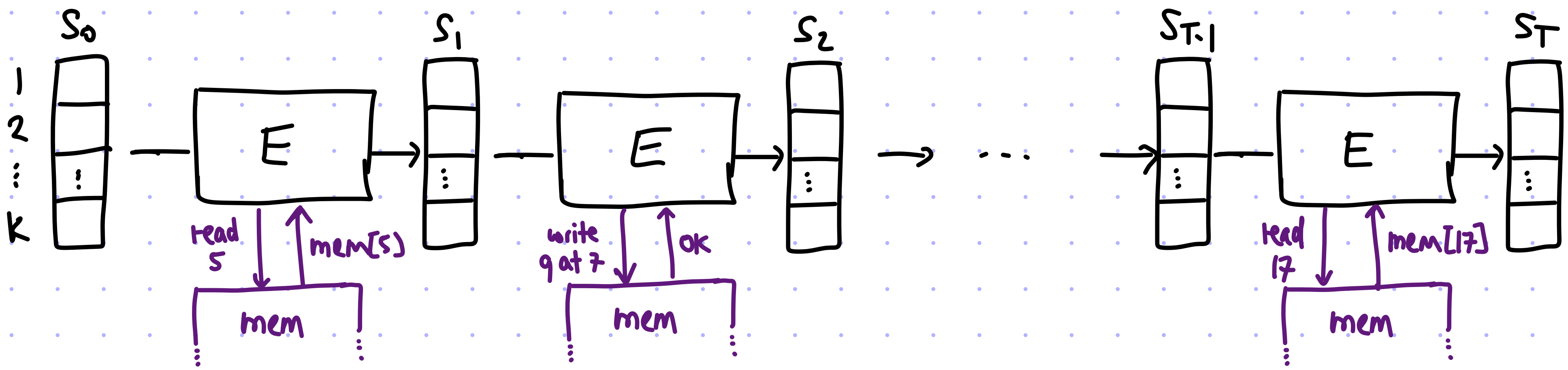
- set proximity parameter $\delta = O\left(\frac{1}{2k+\ell}\right) = O\left(\frac{1}{|E|}\right)$

  $\rightarrow$ this requires setting repetition parameter t in FRI to $t = O(|E|)$ to ensure that $\varepsilon_{LDT}(\delta) = O(1)$
- repeat consistency test $t = O(\log|E|)$ times, as the term becomes $\left(\frac{2|H|-2}{|L|} + (2k+\ell)\delta\right)^t$
- send random coefficients to prover & test $\sum_i \alpha_i f_i + \sum_i \beta_i g_i$ instead of individually

  $\rightarrow$ distortion statements imply the error becomes $\frac{2|H|-2}{|L|} + 2\delta$ (due to column distance)

From Automata to Machines

We now add memory:



If we extend the state with all of memory, we end up with (up to) T^2 variables (beyond linear).

Observation: it suffices to check correctness of memory operations, "what you wrote is what you read".

Consider the memory trace ordered first by address and then by time step.

op	addr	time	val (read or written)
r	2	7	13
r	2	19	13
w	2	22	0
r	2	31	0
r	5	1	3
w	5	6	2
w	7	2	9
⋮			

The memory trace is valid iff for every two adjacent pairs

$$(op, addr, time, val), (op', addr', time', val')$$

the following holds:

- if $addr = addr'$ then $time < time'$ and $(op' = r \rightarrow val' = val)$
- if $addr \neq addr'$ then $addr < addr'$

This leads to a language that represents machine computations...

Memory from a Permuted Trace

<u>lemma:</u> There is a polynomial-time reduction R s.t.

- $R(E,z,T)$ outputs quadratic equations $p_1,\dots,p_m \in \mathbb{F}[X_1,\dots,X_{3k+\ell}]$ with $m,\ell = O(|E|)$
- $(E,z,T) \in BH$ iff $\exists$ augmented execution trace $A_1,\dots,A_k,B_1,\dots,B_\ell : H \to \mathbb{F}$
  & permutation $\pi:[T]\to[T]$ such that
  - $\forall t \in \{0,1,\dots,T-1\}$: $\left\{ p_j \begin{pmatrix} A_1(\omega^t),\dots,A_k(\omega^t), A_1(\omega\cdot\omega^t),\dots,A_k(\omega\cdot\omega^t), B_1(\omega^t),\dots,B_\ell(\omega^t) \\ A_1(\omega^{\pi(t)}),\dots,A_k(\omega^{\pi(t)}) \end{pmatrix} = 0 \right\}_{\forall j \in [m]}$
  - $A_1|_{H_{in}} = z$, $A_1(\omega^{T-1}) = 0$

<u>proof:</u> Set $p_1,\dots,p_m$ to be the quadratic equations obtained by translating the transition function & also the logic for "what you wrote is what you read".

<u>Completeness:</u> choose π to be the permutation that reorders the trace by address then time, so that the memory checks pass

<u>Soundness:</u> for any choice of permutation π, either some memory check fails, or the read/write operations are all correct so the transition function is fed the correct values.

IOP for Algebraic Machines

The reduction in the prior slide directly leads to an IOP for algebraic machines with similar parameters as for algebraic automata (linear proof length, ...) provided we have

IOP Protocol for Vector Permutation Check

Consider the setting where the verifier has oracle access to $f_1, \dots, f_k, g_1, \dots, g_k : L \to \mathbb{F}$ of degree $\leq d$ and wishes to check the claim "$\exists \pi : H \to H$ s.t. $\forall i \in [k]\ \forall a \in H\ \hat{g}_i(a) = \hat{f}_i(\pi(a))$". ⊛

A common technique in the PCP literature relies on routing networks, but they have size $\Omega(T \cdot \log T)$.

In the IOP model we can use interaction to achieve proof length $O(T)$.

See worksheet. For $k=1$, the main idea is to base the protocol on the fact

⊛ is equivalent to asking if $\{\hat{g}(a)\}_{a \in H}$ and $\{\hat{f}(a)\}_{a \in H}$ equal as multisets, which in turn is true iff $\prod_{a \in H} (X - \hat{g}(a)) \equiv \prod_{a \in H} (X - \hat{f}(a))$.