Lecture B.1

Intro to PCPs

(Probabilistically Checkable Proofs)

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MSRI Summer Graduate School on
Foundations and Frontiers of Probabilistic Proofs
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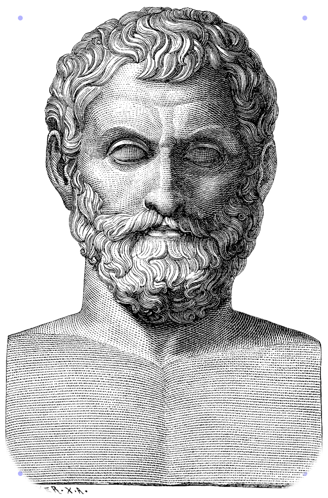
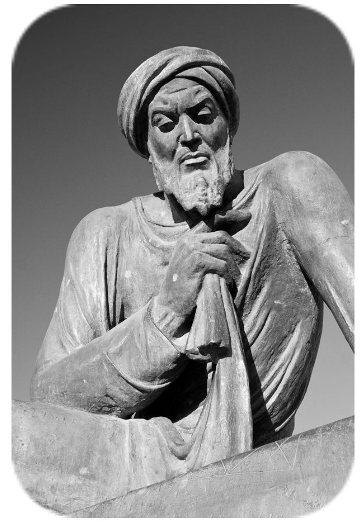
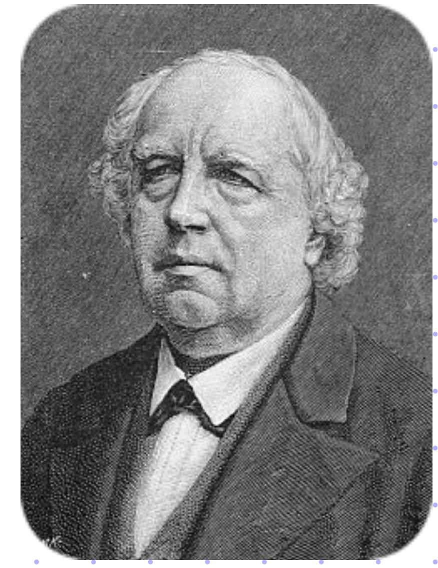
Evolution of mathematical proofs



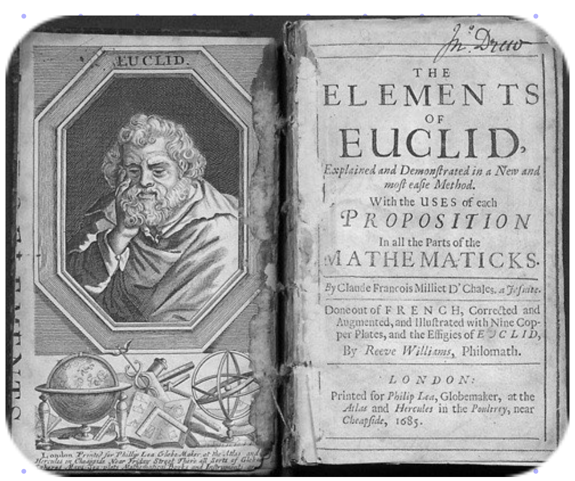
Thales of Miletus



al-Khwarizmi



Weierstrass



Euclid

Madhava of Sangamagrama

Hilbert

Checking a proof without reading it?


Science Times

The New York Times

Biologists Tell a Tale Of Interfering In-Laws

Asteroid Defense: 'Risk Is Real,' Planners Say

New Short Cut Found For Long Math Proofs

A proof can be tested by checking just a part, inventors say.

Proofs have many interpretations:

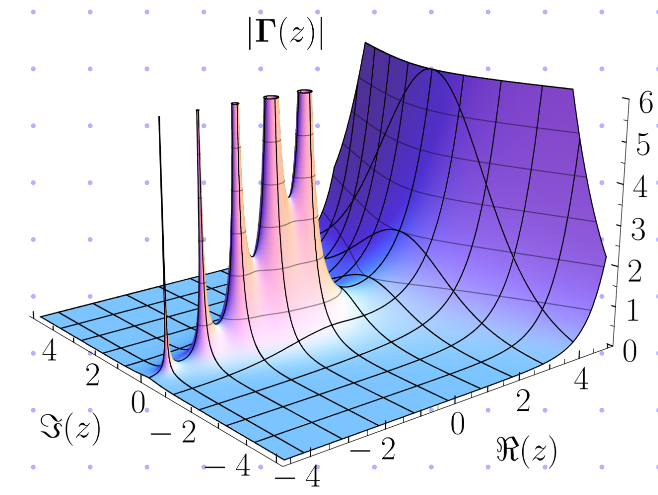
- "logical derivations from axioms" - Zermelo
- "approximation of understanding" - Dinur
- "Whatever that convinces me" - Even

Key idea: check proofs probablistically allowing negligable error

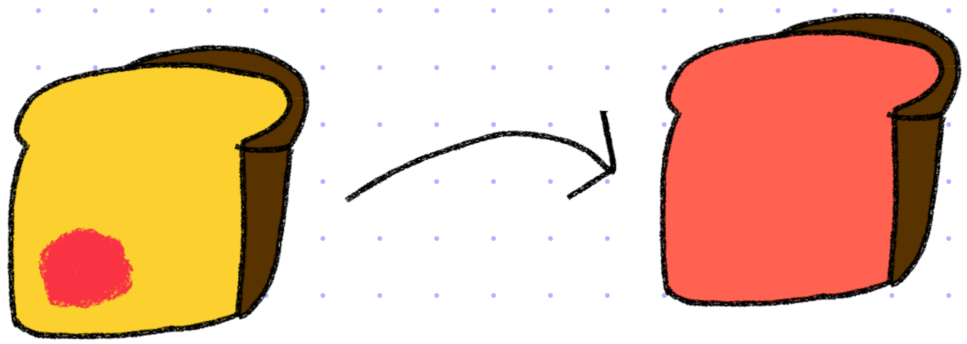
Caveat: What if only one line of the proof is wrong?

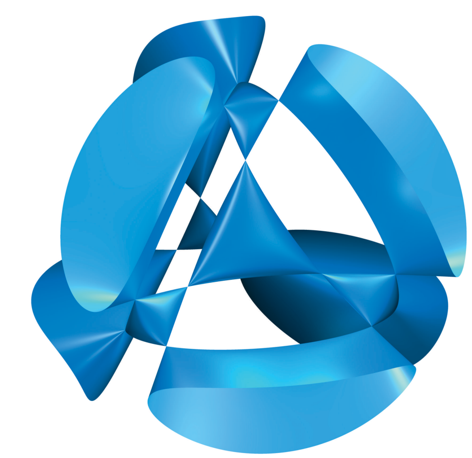
Local-to-global phenomena

Idea: endow proof with a rich structure that allows checking global properties via local constraints!



aka, the "Jam principle"



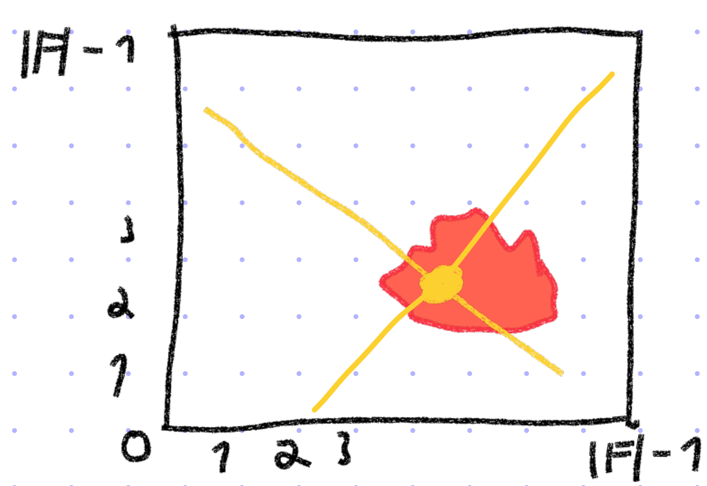


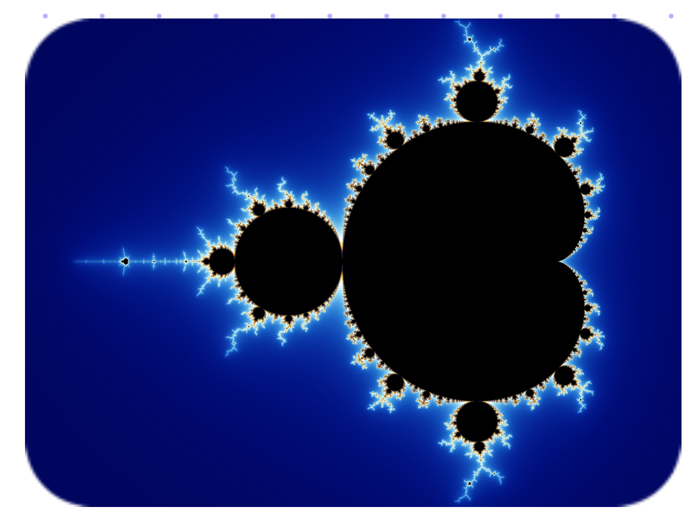
Informal example: low-degree polynomials

$p \in \mathbb{F}[X,Y]$, $\deg(p) \le d$

$p_\ell(t) := p(\bar{x} + t\bar{y}) \in \mathbb{F}[X]$

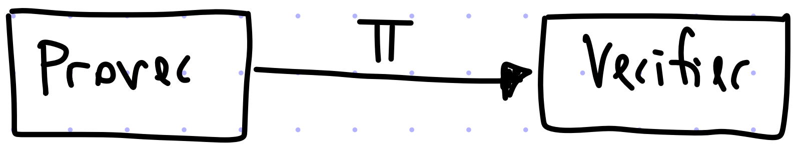
$\deg(p_\ell) \le d$



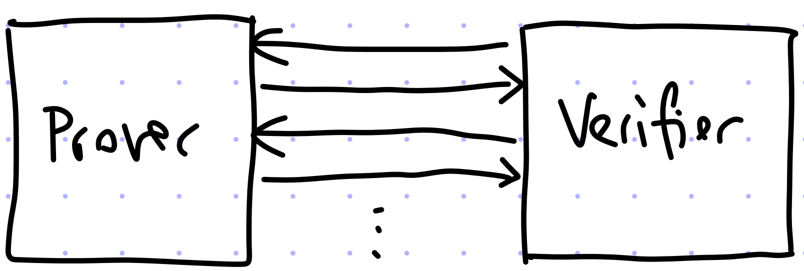


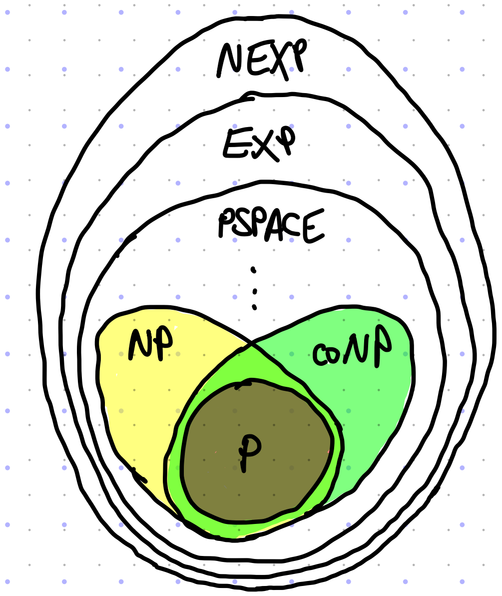
A New Model: Probabilistically Checkable Proofs

- NP represents proofs having a deterministic polynomial-time verifier



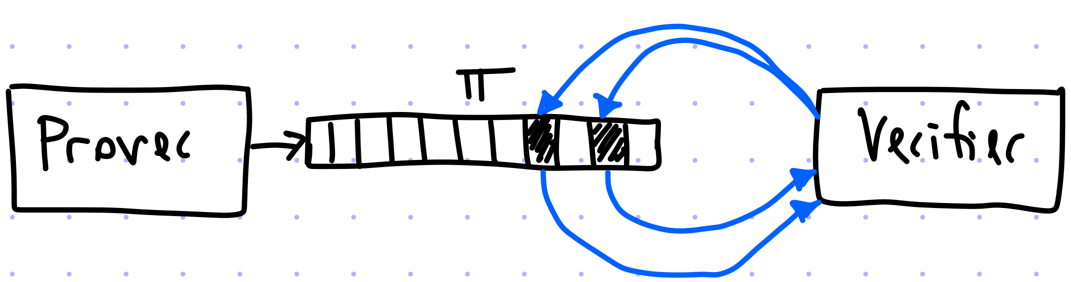
- IP represents proofs where the polynomial-time verifier has two new resources:
  ① randomness , and ② interaction





Today we study a new model:

- PCP represents proofs where the polynomial-time verifier has two new resources:
  ① randomness , and ② oracle access to proof



Definition of PCP

Let P be an all-powerful prover and V a ppt oracle algorithm.
We say that (P,V) is a PCP system for a language L with completeness error ε_c and soundness error ε_s if the following holds:

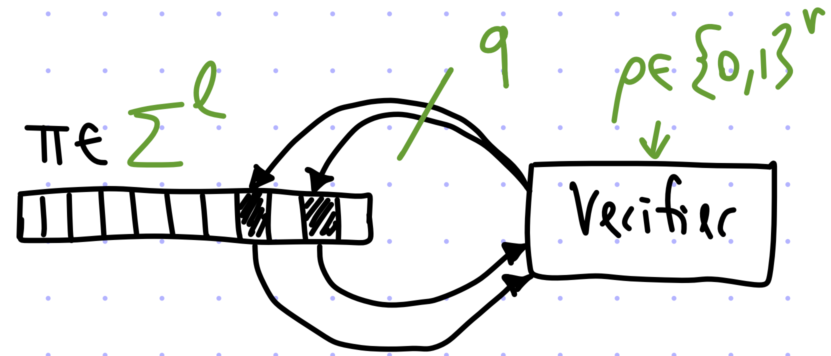
① completeness: $\forall x \in L$, for $\pi := P(x)$, $\Pr_{\rho}[V^{\pi}(x;\rho)=1] \geq 1-\varepsilon_c$

② soundness: $\forall x \notin L\ \forall \tilde{\pi}\ \Pr_{\rho}[V^{\tilde{\pi}}(x;\rho)=1] \leq \varepsilon_s$

We call π a "PCP", and can view it as a "robust encoding" of a witness, which admits verification without reading all its symbols.

For IPs we cared about: round complexity, communication complexity, ...
For PCPs we have a somewhat different set of parameters:

- Σ : proof alphabet
- ℓ : proof length
- q : verifier query complexity
- r : verifier randomness complexity



[typically queries to π will be non-adaptive]

Some Special Cases

We wish to understand $PCP[\varepsilon_c, \varepsilon_s, \Sigma, \ell, q, r, \ldots]$ in different regimes.
Let's start with some special cases to warm up.

Suppose there is no proof $(q=0)$:

- $PCP[q=0, r=0] = P$ ← if there is no proof and no randomness then the verifier is just a polytime algorithm
- $PCP[q=0, r=O(\log n)] = P$ ← logarithmically-many random bits don't help
- $PCP[q=0, r=poly(n)] = BPP$ ← if there is randomness but no proof then the verifier is just a ppt algorithm

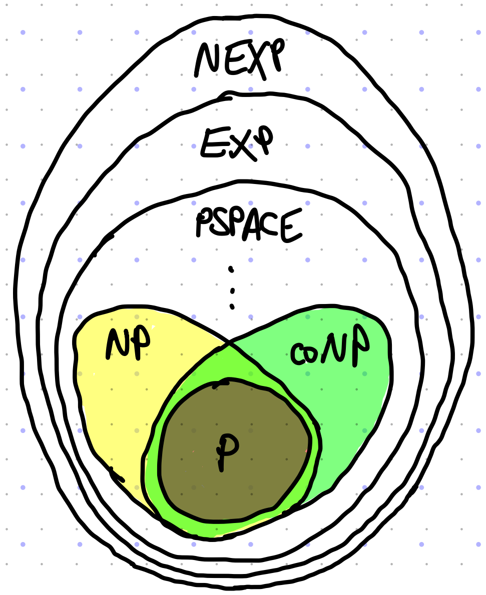
Suppose there is no randomness $(r=0)$:

- $PCP[q=poly(n), r=0] = NP$ ← verifier can read in full a poly-size witness

We denote by PCP the complexity class with no restrictions beyond "V is ppt".
This means that $q=poly(n)$, $r=poly(n)$ and allows for $\ell = exp(n)$, $|\Sigma| = exp(n)$.

Questions

- Which languages have PCPs (beyond NP & BPP)? — more than PSPACE
- Do PCPs have benefits for NP languages? — yes
  (E.g. query complexity sublinear in witness size)
- Do PCPs have benefits for tractable languages? — yes
  (E.g. PCP verification faster than execution)
- Are there ZK PCPs for NP languages? — yes

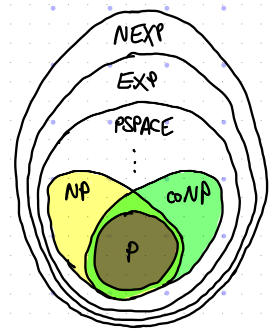


Many good news!

But the PCP model is weird (PCP verifier has oracle access to a large proof).
How are PCPs useful?

① lead to interactive arguments (& other crypto proofs) with strong efficiency features
② lead to hardness & approximation results

Upper Bound on PCPs



- Σ : proof alphabet
- ℓ : proof length
- q : verifier query complexity
- r : verifier randomness complexity

theorem: $PCP \subseteq NEXP$

lemma: (i) $\ell \leq 2^r q$ for non-adaptive verifiers
(ii) $\ell \leq 2^r |\Sigma|^q q$ for adaptive verifiers

[in constructions ℓ is usually smaller than these upper bounds]

proof of (i): there are at most 2^r different query sets
proof of (ii): each answer from the proof can lead to a different next query

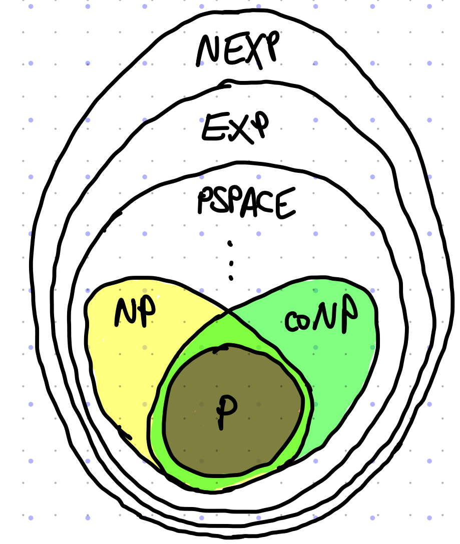
lemma: $PCP[\ell, r] \subseteq NTIME\left((2^r+\ell)\cdot poly(n)\right)$.

proof: Suppose (P,V) is a PCP system for L where the PCP verifier uses r random bits to query a proof of length ℓ. Consider this decider:

$D(x, \pi) :=$ For every $\rho \in \{0,1\}^r$ compute $b_\rho := V^\pi(x;\rho)$ and output 1 iff $\sum_\rho b_\rho / 2^r \geq 1-\varepsilon_c$.
($x \in \{0,1\}^n$, $\pi \in \{0,1\}^\ell$)

If $x \in L$ then $\exists \pi$ s.t. $D(x,\pi)=1$. If $x \notin L$ then $\forall \pi$ $D(x,\pi)=0$. ∎

A Simple Inclusion: PSPACE



theorem: PSPACE $\subseteq$ PCP

lemma: IP $\subseteq$ PCP

proof: Suppose that (P,V) is a public-coin IP for L. (Public coin comes WLOG.)
Consider proofs in this format: $\Pi = \{a_{r_1}\}_{r_1} \cup \{a_{r_1,r_2}\}_{r_1,r_2} \cup \cdots \cup \{a_{r_1,\dots,r_k}\}_{r_1,\dots,r_k}$.
The PCP verifier samples $r_1,\dots,r_k$ and accepts if the IP verifier accepts:

$$V(x, a_{r_1}, a_{r_1,r_2}, \dots, a_{r_1,\dots,r_k}; r_1 \dots r_k) \stackrel{?}{=} 1.$$

Completeness: consider the honest proof $\Pi := \{P(x,r_1)\}_{r_1} \cup \{P(x,r_1,r_2)\}_{r_1,r_2} \cup \cdots \cup \{P(x,r_1,\dots,r_k)\}_{r_1,\dots,r_k}$.
Soundness: any proof in the above format corresponds to an "unrolled" IP prover ■

In sum: PSPACE $\subseteq$ PCP $\subseteq$ NEXP. We will see that PCP = NEXP by recycling techniques (arithmetization, sumcheck) and using new ones (low-degree testing).
We will also see how to "scale down" to get PCPs for NP.

Course outline

Local-to-global phenomena

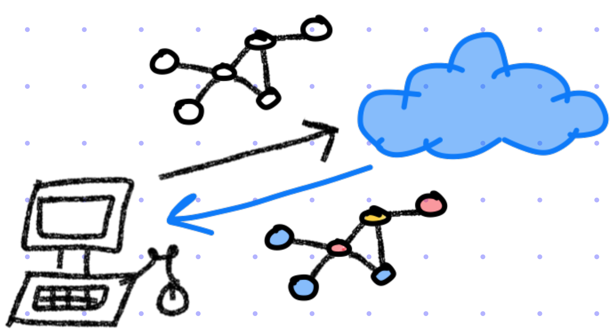
- Linearity testing
- Low-degree testing
- FFT-based testing of univariate polynomials

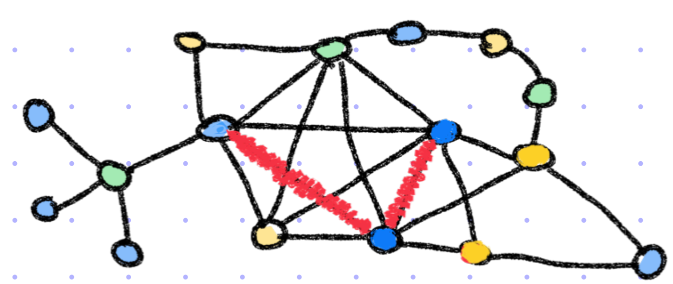
PCP constuctions

- exp-size, $O(1)$-local PCPs
- poly-size, polylog-local PCPs
- PCP composition
- Sublinear-time verification

Applications

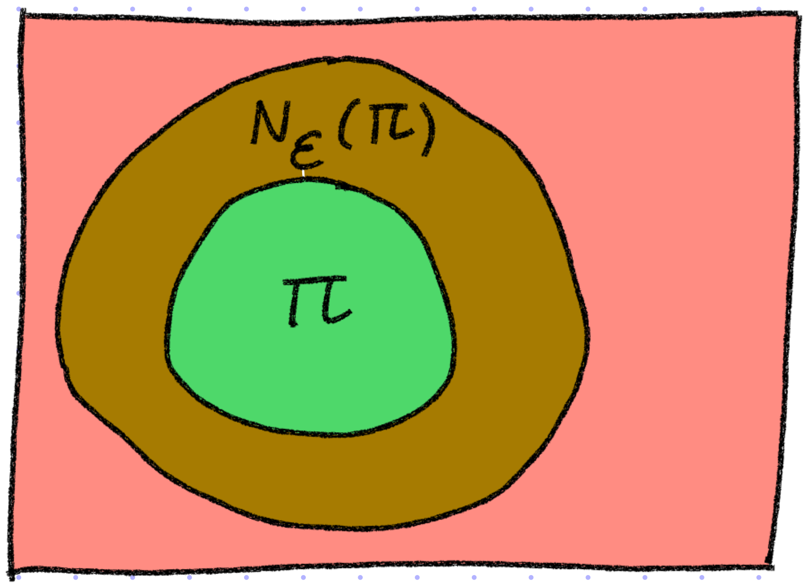
- Delegation of computation
- Hardness of approximation





Conceptual perspectives

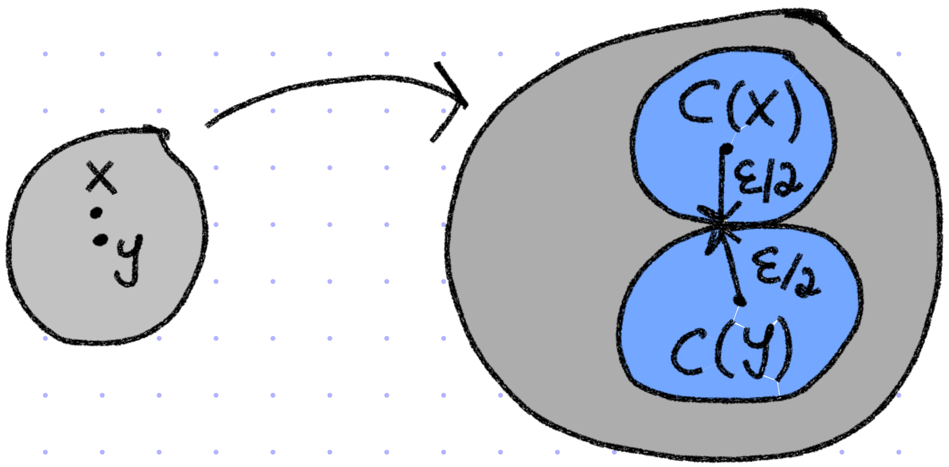
Property testing



Is $f \in \Pi$ (e.g. $\Pi = \mathbb{F}^{\leq d}[X]$)

or $\delta(f, \Pi) > \varepsilon$?

Coding theory



For every $x \neq y$, we have

$\delta(C(x), C(y)) > \varepsilon$

e.g., univariate polynomials

low-degree polynomials

Linear func. on hypercube