Foundations and Frontiers of Probabilistic Proofs (Summer 2021)
Worksheet A.3: IPs for PSPACE
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We work out Shamir's original proof that $\mathsf{IP} = \mathsf{PSPACE}$, which relies on fully quantified boolean formulas *with a special structure*. We say that a fully quantified boolean formula is *simple* if every occurrence of every variable is separated from its quantification point by at most one universal quantifier ($\forall$) and arbitrarily many other symbols.

- This is a simple formula: $\forall x_1 \forall x_2 \exists x_3 \left((x_1 \vee x_2) \wedge x_3\right)$.
  What is its value?
- This formula is not simple: $\exists x_1 \forall x_2 \left((x_1 \vee x_2) \wedge \forall x_3 (x_1 \vee x_3)\right)$.
  What is its value?

We denote by TSQBF language obtained by considering only fully quantified boolean formulas that are simple.

Problem 1. Which of the following fully quantified boolean formulas are simple? What is their value?

1. $\forall x_1 \, \forall x_2 \, \exists x_3 \left((x_1 \wedge x_2 \wedge x_3) \wedge \forall x_4 \, (\neg x_1 \wedge x_4)\right)$
2. $\forall x_1 \, \forall x_2 \, \exists x_3 \left((x_1 \vee x_3) \wedge \forall x_4 \, (x_3 \vee (x_2 \wedge x_4))\right)$
3. $\forall x_1 \left(\exists x_2 \forall x_3 (x_1 \vee x_2 \vee \neg x_3) \wedge \forall x_4 \, (\neg x_1 \wedge x_4)\right)$
4. $\forall x_1 \left(\exists x_2 \forall x_3 (x_1 \vee x_2 \vee \neg x_3) \wedge \exists x_4 \, (\neg x_1 \wedge x_4)\right)$

Problem 2. In this question we prove that a fully quantified boolean formula can be efficiently transformed into a simple fully quantified boolean formula with the same value. The general idea is to define a fresh variable for each occurrence of each variable in the original formula.

Let Φ be a fully quantified boolean formula with variables $x_1, \ldots, x_n$. We define a new formula Ψ that has a variable for each universal quantifier crossed by each variable in Φ. For example, if x_1 crosses k universal quantifiers in Φ, then Ψ has variables $x_{1,1}, \ldots, x_{1,k}$.

1. Give a boolean formula which is true if and only if x_1 and x_2 are equal.
2. Let $\Phi = \exists x_1 \forall x_2 \left((x_1 \vee x_2) \wedge \forall x_3 (x_1 \vee x_3)\right)$. By replacing the two occurrences of x_1 with $x_{1,1}$ and $x_{1,2}$, and adding constraints and quantifiers, obtain a simple formula Ψ that has the same value as Φ.
3. Give an efficient algorithm that transforms a QBF Φ into an equisatisfiable simple QBF Ψ.
4. Prove the algorithm's correctness.

Problem 3. Outline an interactive proof for TSQBF. *Hint: show that simple formulas have "nice" arithmetizations.*