

6. EXERCISE LIST. MONDAY WEEK 2

Exercise 6.1. Let $f := \sum_{\alpha \in A} b_{\alpha} \mathbf{x}^{\alpha}$ be a polynomial for some finite $A \subseteq \mathbb{N}^n$.

- Prove that if the coefficient of one vertex is negative, then f attains negative values over $\mathbb{R}_{>0}^n$.
- Show that if f is nonnegative over $\mathbb{R}_{>0}^n$ then for all $\alpha \in \text{vert}(\text{New}(f))$ it holds that $b_{\alpha} > 0$.
- Is it true that if there exist two vertices of $\text{vert}(\text{New}(f))$ with opposite signs, there exists $x \in \mathbb{R}_{>0}^n$ such that $f(x) = 0$?

Exercise 6.2. Consider the polynomial $p = 1 - 4xy^3 + x^2y^3 + 3x^2y$. Draw the Newton polytope $\text{NP}(f)$ and find the outer normal fan (the set of outer normal cones of all faces of $\text{NP}(f)$). Use the cones to find systematically two points $a, b \in \mathbb{R}_{>0}^n$ such that $p(a) > 0$ and $p(b) < 0$ (this exercise builds on the previous one).

Exercise 6.3. A multiaffine polynomial in $\mathbb{R}[x_1, \dots, x_n]$ is a polynomial such that all exponents are either 1 or 0. For example, $f = x_1x_2 + x_1x_3 + x_1x_2x_3 \in \mathbb{R}[x_1, x_2, x_3]$ is multiaffine.

Show that if f is multiaffine, then all exponent vectors are vertices of its Newton Polytope. Conclude that the polynomial has a root in $\mathbb{R}_{>0}^n$ if and only if it has coefficients of opposite signs.

Note: This is used to prove the (det) statement in the injectivity theorem.

Exercise 6.4. Let $f = \sum_{i=1}^k s_i^2 \in \mathbb{R}[\mathbf{x}]$ be a sum of squares. Show that we have the following equality of the involved Newton polytopes:

$$\text{New}(f) = \text{conv} \left(\bigcup_{i=1}^k 2 \cdot \text{New}(s_i) \right).$$

Exercise 6.5. The *Motzkin polynomial*

$$M(x_1, x_2) := 1 + x_1^2x_2^4 + x_1^4x_2^2 - 3x_1^2x_2^2 \in \mathbb{R}[x, y]$$

was the first known explicit example of a polynomial that is nonnegative but not a sum of squares. Show both of these properties.

Hints:

- (1) To show nonnegativity use the classical *Arithmetic-Geometric Mean Inequality*, which is stating: for $t_1, \dots, t_d \in \mathbb{R}_{\geq 0}$ it holds that

$$\frac{1}{d} \sum_{i=1}^d t_i \geq \sqrt[d]{\prod_{i=1}^d t_i}$$

with equality if and only if $t_1 = t_2 = \dots = t_d$.

- (2) To show that M is not a sum of squares, do a term by term inspection: investigate how a negative term could be created, use the result from Exercise 6.4, and show that M being SOS leads to a contradiction.

Exercise 6.6. A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is called a *circuit polynomial* if it is of the form

$$f(\mathbf{x}) = \sum_{j=0}^n c_{\alpha(j)} \mathbf{x}^{\alpha(j)} + c_{\beta} \mathbf{x}^{\beta}$$

with support $A = \{\alpha(0), \dots, \alpha(n), \beta\} \subset \mathbb{N}^n$, such that

- (1) the coefficients satisfy $c_{\alpha(j)} \in \mathbb{R}_{>0}$ for all $0 \leq j \leq n$, and $c_{\beta} \in \mathbb{R}$,
- (2) $\text{New}(f)$ is a simplex with even vertices, which are given by the $\alpha(j)$, and
- (3) the exponent β is in the strict interior of $\text{New}(f)$.

It can be shown that a circuit polynomial is nonnegative over $\mathbb{R}_{>0}^n$ if and only if

$$-c_{\beta} \leq \prod_{j=0}^n \left(\frac{c_{\alpha(j)}}{\lambda_j} \right)^{\lambda_j}$$

where λ_j are the unique barycentric coordinates of β with respect to the $\alpha(j)$.

- (1) Using the theorem above, verify again that the Motzkin polynomial is nonnegative over $\mathbb{R}_{>0}^n$.
- (2) Investigate now the following variation of the Motzkin polynomial:

$$M_2(x_1, x_2) := 1 + 2 \cdot x_1^2 x_2^4 + 3 \cdot x_1^4 x_2^2 - a \cdot x_1^2 x_2^2 \in \mathbb{R}[x, y],$$

with $a \in \mathbb{R}_{>0}$. What is the maximal a , such that M_2 is nonnegative?

Exercise 6.7. Go through all the networks in the exercises of the previous week. Check which of them are MESSI and in this case, check if the conditions for the validity of the theorems we presented in the class are valid (and thus their statements hold :-)).