INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY MSRI, SUMMER 2023 MINICOURSE ON DG CATEGORIES

DIMA ARINKIN

Problem set 2
Derived category of modules. Semi-free resolutions

(Problems marked with $*$ are slightly harder, or perhaps they concern some side topics, so I consider them less important.)

Everywhere below, R is a commutative ring (or rather a $\Bbbk$-algebra).

1*. Let $x, y \in C(R)$ be any two complexes, $\phi : x \to y$ and $\psi : y \to x$ be two morphisms such that $\psi \circ \phi$ is homotopic to the identity map. Show that we can replace y by a quasi-isomorphic complex y' such that x is isomorphic to a direct summand of y'. (You can take y' to be the mapping cylinder of ϕ.)

2. Prove that the following definitions of a perfect complex $x \in C(R)$ are equivalent:

(1) x is quasi-isomorphic to a direct summand of a bounded complex of free R-modules of finite rank;
(2) x is a direct summand of a complex that is quasi-isomorphic to a bounded complex of free R-modules of finite rank;
(3) x is obtained from the free module R by a finite sequence of taking cohomological shifts, taking cones, taking direct summands, and replacing a complex by a quasi-isomorphic complex.

3. (colimits provide direct summands) Consider $\mathcal{A} = C(R)$ (or, really, any category with colimits and cones). Suppose x is a direct summand of y. Write x as a colimit

$$y \to y \to \dots$$

for appropriate choice of maps.

4. (cones and direct sums provide direct summands) Continuing with the previous problem, rewrite x as the cone of a morphism

$$\bigoplus_{i=0}^{\infty} y \to \bigoplus_{i=0}^{\infty} y.$$

in $\mathcal{A}$.

Remark. More generally, you can rewrite any colimit indexed by $\mathbb{Z}^+$ using cones and direct sums. This idea allows us to define such colimits (or, more properly, homotopy colimits) in a triangulated category. (I first saw this in Böxstedt-Neeman's paper.)

5*. (Bar resolution continued) Let $B_\bullet$ be a resolution of R by free R-bimodules. Given any complex $M^\bullet \in K(R)$, consider the bicomplex $B_\bullet \otimes M^\bullet$. Show that the total complex of this bicomplex is a semifree resolution of M.