INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY MSRI, SUMMER 2023 MINICOURSE ON DG CATEGORIES

DIMA ARINKIN

Problem set 1
DG Categories and DG functors

(Problems marked with $*$ are slightly harder, or perhaps they concern some side topics, so I consider them less important.)

1. Consider the DG category of complexes of R-modules $C(R)$. Verify that composition of homomorphisms defines a morphism of complexes

$$\mathrm{Hom}(C_2^\bullet, C_3^\bullet) \otimes \mathrm{Hom}(C_1^\bullet, C_2^\bullet) \to \mathrm{Hom}(C_1^\bullet, C_3^\bullet)$$

2. Let $\mathcal{A}$ be a DG category. Prove that a morphism $\phi : x \to y$ in $\mathcal{A}$ is a quasi-isomorphism if and only if for any object $z \in \mathcal{A}$, the induced map

$$\mathrm{Hom}(z, x) \to \mathrm{Hom}(z, y)$$

is a quasi-isomorphism. (It is natural to reduce this to a form of Yoneda's Lemma, but doing this rigorously requires a bit of caution.

3. (From Toën's notes) Let R be the (classical) ring $k[x]/(x^2)$. Consider the (commutative) DG algebra R', which, as a graded algebra, equals $k[x, y]$ with $\deg(x) = 0$, $\deg(y) = -1$, and d is defined via $dx = 0$, and $dy = x^2$. Now consider R and R' as DG categories with a single object.

Prove that there is a quasi-equivalence between these categories, but only in one direction.

4*. Let us consider the simplest example of localization (or, rather, quotient) of a DG category. Let $\mathcal{A}$ be a DG category with an object $x \in \mathcal{A}$. (Feel free to assume that $\mathcal{A}$ consists of two objects.) Construct a new DG category $\mathcal{B} = \mathcal{A}/\{x\}$ characterized by the following universal property: there is a universal DG functor $\phi : \mathcal{A} \to \mathcal{B}$ together with a contracting homotopy of $\phi(x)$ (that is, a degree -1 element in the complex $\mathrm{Hom}(\phi(x), \phi(x))$ whose derivative is $id_{\phi(x)}$. (The description should be sufficiently explicit: e.g., what is the complex $\mathrm{Hom}_{\mathcal{B}}(\phi(x), \phi(x))$?)

(This is a special case of Drinfeld's construction of quotients of DG categories.)

5*. (Bar resolution) Let R be a ring. Let $B_\bullet$ be a resolution of R by projective R-bimodules; there is a standard such resolution called the *bar resolution*:

$$\dots R \otimes R \otimes R \to R \otimes R \to R.$$

(If you haven't seen how to write the morphisms here, try guessing or look it up!)

Show that for any two R-modules M and N, the complex

$$\mathrm{Hom}_{R \otimes R^{op}}(B_\bullet, \mathrm{Hom}_k(M, N))$$

computes $\mathrm{Ext}^\bullet_R(M, N)$.

(There are at least two natural ways of thinking about this problem: via the composition of derived functors and via natural resolutions of modules.)