DAG Exercises - I - Functor of points

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I.1. Let k be a field and let $X = \mathbf{A}^4_k = \operatorname{Spec} k[x,y,z,w]$. Let Y be the union of the planes $\{x=0, y=0\}$ and $\{z=0, w=0\}$ and let W be the plane $\{x=z, y=w\}$. Write down the coordinate rings of Y and W. Compute $\operatorname{Tor}_i^{\mathcal{O}(X)}(\mathcal{O}(Y), \mathcal{O}(W))$ for all i. Compute

$$\sum_i (-1)^i \operatorname{len}_k(\operatorname{Tor}_i^{\mathcal{O}(X)}(\mathcal{O}(Y), \mathcal{O}(W))).$$

I.2. Let $\mathcal{C}$ be a (small) category and let $\mathcal{P}(\mathcal{C})^{\heartsuit} = \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$.

(a) Show that limits and colimits in $\mathcal{P}(\mathcal{C})^{\heartsuit}$ are computed pointwise in the following sense: for any object $X \in \mathcal{C}$ and any diagram $F \colon I \to \mathcal{P}(\mathcal{C})^{\heartsuit}$, the limit and colimit $\lim_{i\in I} F_i$ and $\operatorname{colim}_{i\in I} F_i$ exist and the natural maps $(\lim_{i\in I} F_i)(X) \to \lim_{i\in I}(F_i(X))$ and $\operatorname{colim}_{i\in I}(F_i(X)) \to (\operatorname{colim}_{i\in I} F_i)(X)$ are isomorphisms.
(b) Show that the Yoneda embedding $h \colon \mathcal{C} \to \mathcal{P}(\mathcal{C})^{\heartsuit}$ preserves any limits that exist in $\mathcal{C}$.
(c) Find an example of a category $\mathcal{C}$ which has finite colimits but where $h \colon \mathcal{C} \to \mathcal{P}(\mathcal{C})^{\heartsuit}$ does not preserve finite colimits.

I.3. Find necessary and sufficient conditions for $\operatorname{Spec} R$ to represent a functor $\mathrm{Aff}_k \to \mathrm{Set}$ valued in groups. What about in commutative groups? The answer should be expressed in terms of extra structures on R.

I.4. Write down the structures you found in Exercise I.3 in the case of the group schemes $\mathbf{G}_a$, $\mathbf{G}_m$, μ_n, and α_p. Here, for $R \in \mathrm{CAlg}_{\mathbf{Z}}$,

(i) $\mathbf{G}_a(R) = R$ as a group under addition,
(ii) $\mathbf{G}_m(R) = R^{\times}$ as a group under multiplication, and
(iii) $\mu_n(R) = \{u \in R : u^n = 1\}$,

while for $R \in \mathrm{CAlg}_{\mathbf{F}_p}$

(iv) $\alpha_p(R) = \{x \in R : x^p = 0\}$.

I.5. Prove that in characteristic p the schemes μ_p and α_p are isomorphic, but not as group schemes.

I.6. Recall that $\mathbf{P}^1$ is covered by two copies of $\mathbf{A}^1$ which intersect in $\mathbf{G}_m$. Show that the commutative square

$$\begin{array}{ccc} \mathbf{G}_m & \longrightarrow & \mathbf{A}^1 \\ \downarrow & & \downarrow \\ \mathbf{A}^1 & \longrightarrow & \mathbf{P}^1 \end{array}$$

is a pushout square in $\mathrm{Shv}_\tau(\mathrm{Aff}_k)^\heartsuit$ but not in $\mathcal{P}(\mathrm{Aff}_k)^\heartsuit$. Let P denote the pushout in $\mathcal{P}(\mathrm{Aff}_k)^\heartsuit$. Show that the τ-sheafification of P is $\mathbf{P}^1$.