DAG Exercises - 2 - Axiomatic (lazy) ∞-category theory

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II.1. Let $\mathcal{C} = F_\bullet$ be a quasi-category. Define a category $\pi\mathcal{C}$ as follows. The objects are the elements of F_0. The morphisms should be certain equivalence classes of elements of F_1; the identity maps should be the degenerate 1-simplices. Define this notion of equivalence and composition (using F_2), showing that with your definition $\pi\mathcal{C}$ is indeed a category. This is a model for the homotopy category of $\mathcal{C}$.

II.2. Let $A_\bullet$ be a simplicial abelian group. Show that $d : A_n \to A_{n-1}$ defined as $\sum_{i=0}^{n}(-1)^i\partial_i$ makes

$$0 \leftarrow A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$$

into a chain complex of abelian groups.

II.3. If $\mathcal{C}$ is a category and $\mathrm{N}_\bullet\mathcal{C}$ denotes its nerve, show that $\mathrm{N}_\bullet\mathcal{C}$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

II.4. Recall that a Kan complex is said to be contractible if every square

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & * \end{array}$$

admits a dotted arrow making the diagram commutative. If $\mathcal{C}$ is a category, what does it mean for $\mathrm{N}_\bullet\mathcal{C}$ to be a contractible Kan complex?

References