INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY
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MINICOURSE ON DG CATEGORIES

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Problem set 3
Modules over a DG category

(Problems marked with $*$ are slightly harder, or perhaps they concern some side topics, so I consider them less important.)

Everywhere below, $\mathcal{A}$ is a DG category, and $C(\mathcal{A})^{sf} \subset C(\mathcal{A})$ is the full subcategory of semifree objects.

1. (Example of twisted complexes) Let R be a DG ring, and let $M \in C(\mathcal{A})$ be a semifree module "of rank 3": it admits a filtration with quotients M_0, M_1/M_0, M_2/M_1 being free of rank one. Apply the strategy from the lecture: Lift generators of these free modules to elements e_0, e_1, e_2 of M, and write the conditions on the differential $d(e_i)$ in this basis. Do not assume that e_i's have degree 0.

2. Let $\mathcal{A}$ be a DG category with finite set of objects $x_1, \ldots, x_k$. Show that the category $C(\mathcal{A})$ is equivalent to the category $C(R)$ for some DG algebra R. Is this a strict equivalence, or only a quasi-equivalence?

(You can take $R = \bigoplus_{i,j} \operatorname{Hom}_{\mathcal{A}}(x_i, x_j)$.)

The 'clever' way to do solve this is to use the free object $\bigoplus_i h_{x_i} \in C(\mathcal{A})$, but it can also be done directly.

What happens if some or all of x_i's are isomorphic?

3. Continuing with the previous problem: under the equivalence $C(\mathcal{A}) \simeq C(R)$, show that any semifree R-module is semifree as an $\mathcal{A}$-module, and that a semifree $\mathcal{A}$-module is isomorphic to a direct summand of a semifree R-module.

4*. Continuing with the previous question: show that the embedding $C(R)^{sf} \to C(\mathcal{A})^{sf}$ is a quasi-equivalence.

5*. (An easier version of a question from lecture) Suppose $M \in C(\mathcal{A})$ has a subcomplex N such that N is semifree and M/N is *free*. Show that M is semifree. (You can also try the harder statement where you only suppose that M/N is semifree.)