# INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY
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# MINICOURSE ON DG CATEGORIES

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## Problem set 2
## Derived category of modules. Semi-free resolutions

(Problems marked with $*$ are slightly harder, or perhaps they concern some side topics, so I consider them less important.)

Everywhere below, $R$ is a commutative ring (or rather a $\Bbbk$-algebra).

**1**$^*$. Let $x, y \in C(R)$ be any two complexes, $\phi : x \to y$ and $\psi : y \to x$ be two morphisms such that $\psi \circ \phi$ is homotopic to the identity map. Show that we can replace $y$ by a quasi-isomorphic complex $y'$ such that $x$ is isomorphic to a direct summand of $y'$. (You can take $y'$ to be the mapping cylinder of $\phi$. If you don't remember what the mapping cylinder is, this is a good excuse to look it up or ask someone.)

**2.** Prove that the following definitions of a perfect complex $x \in C(R)$ are equivalent:

(1) $x$ is quasi-isomorphic to a direct summand of a bounded complex of free $R$-modules of finite rank;
(2) $x$ is a direct summand of a complex that is quasi-isomorphic to a bounded complex of free $R$-modules of finite rank;
(3) $x$ is obtained from the free module $R$ by a finite sequence of taking cohomological shifts, taking cones, taking direct summands, and replacing a complex by a quasi-isomorphic complex.

**3.** (colimits provide direct summands) Consider $\mathcal{A} = C(R)$ (or, really, any category with colimits and cones). Suppose $x$ is a direct summand of $y$. Write $x$ as a colimit

$$y \to y \to \dots$$

for appropriate choice of maps.

**4.** (cones and direct sums provide direct summands) Continuing with the previous problem, rewrite $x$ as the cone of a morphism

$$\bigoplus_{i=0}^{\infty} y \to \bigoplus_{i=0}^{\infty} y.$$

in $\mathcal{A}$.

*Remark.* More generally, you can rewrite any colimit indexed by $\mathbb{Z}^+$ using cones and direct sums. This idea allows us to define such colimits (or, more properly, homotopy colimits) in a triangulated category. (I first saw this in Böxstedt-Neeman's paper.)

**5**$^*$. (Bar resolution continued) Let $B_\bullet$ be a resolution of $R$ by free $R$-bimodules. Given any complex $M^\bullet \in K(R)$, consider the bicomplex $B_\bullet \otimes M^\bullet$. Show that the total complex of this bicomplex is a semifree resolution of $M$.