## DAG Exercises - 3 - Flavors of derived commutative rings

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Summer 2023

**III.1.** Show that the Day convolution symmetric monoidal structure on  $\operatorname{GrMod}_k^{\otimes D}$  described in class is indeed a symmetric monoidal category. Precisely, it should satisfy the axioms of a tensor category as layed out in [1, Sec. II.1]. Show the same thing for the Koszul symmetric monoidal structure  $\operatorname{GrMod}_k^{\otimes \kappa}$ .

**III.2.** Show that  $\mathbf{D}^{\vee} \simeq \mathbf{Z} \oplus \mathbf{Z}(-1)$  admits a unique commutative algebra structure in either  $\operatorname{GrMod}_{k}^{\otimes_{\mathrm{D}}}$  or  $\operatorname{GrMod}_{k}^{\otimes_{\mathrm{K}}}$ .

**III.3.** Prove that  $\mathbf{D}^{\vee}$  admits no bicommutative bialgebra structure in  $\operatorname{GrMod}_{\mathbf{Z}}^{\otimes_{\mathrm{D}}}$ . Prove that it admits a unique such structure in  $\operatorname{GrMod}_{\mathbf{Z}}^{\otimes_{\mathrm{K}}}$ . (Hint: don't forget the counit condition!)

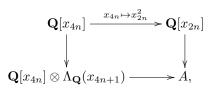
**III.4.** Prove that if  $\mathcal{C}$  is a symmetric monoidal category and  $B \in \operatorname{cCAlg}(\operatorname{CAlg}(\mathcal{C}))$  is a bicommutative bialgebra, then there is a canonical symmetric monoidal structure on  $\operatorname{Mod}_B(\mathcal{C})$  such that the forgetful functor  $\operatorname{Mod}_B(\mathcal{C}) \to \mathcal{C}$  is symmetric monoidal.

**III.5.** Question (I really do not know the answer at the moment): is there a simplicial enrichment for the category of non-negative cdgas over  $\mathbf{Q}$  making it into a simplicial model category. Hint: you could attempt to transport the structure using the free functor from non-negative chain complexes and the Dold–Kan correspondence.

**III.6.** Let  $\mathbf{Q}[x_{2n}]$  be the free cdga on a degree 2n generator for  $n \ge 0$ . Compute  $\pi(\mathbf{Q}[x_{2n}], X)$ , the set of homotopy classes of cdga maps  $\mathbf{Q}[x_{2n}] \to X$ , as a functor in X. (The same argument works for free cdgas on odd generators.)

**III.7.** Compute a factorization of  $\mathbf{Q}[x_0] \to \mathbf{Q}[x_0] \times \mathbf{Q}[x_0]$  as  $\mathbf{Q}[x_0] \to \mathbf{Q}[x_0]^I \to \mathbf{Q}[x_0] \times \mathbf{Q}[x_0]$  where the first map is a trivial cofibration and the second map is a fibration. (Thus,  $\mathbf{Q}[x_0]^I$  is a path object for  $\mathbf{Q}[x_0]$ .)

**III.8.** Consider  $\mathbf{Q}[x_{2n}]/(x_{2n}^2)$  as a cdga with 0 differential where  $x_{2n}$  has degree 2n. On the other hand, consider the cdga A defined as the pushout



where in the bottom left  $d(x_{4n+1}) = x_{4n}$ . Of course, A and  $\mathbf{Q}[x_{2n}]/(x_{2n}^2)$  are weakly equivalent. (Construct a specific quasi-isomorphism!). Complete the following exercises, referring to [2, Sec. 4] for further model categorical details.

- (a) Prove that the cdga A is fibrant and cofibrant in the model category structure on non-negative cdgas over  $\mathbf{Q}$ .
- (b) Prove that the cdga  $\mathbf{Q}[x_{2n}]/(x_{2n}^2)$  is fibrant but not cofibrant.
- (c) Compute  $\pi(A, X)$ , the set of maps  $A \to X$  up to homotopy, as a functor of the cdga X.
- (d) Prove that  $\pi^r(\mathbf{Q}[x_{2n}]/(x_{2n}^2), X)$  cannot agree with  $\pi(A, X)$  by finding an appropriate choice of X.

**III.9.** Is  $\mathbf{Q}[x_0]$  cofibrant when viewed as a cdga? What about  $\mathbf{Q}[x_0^{\pm 1}]$ ?

**III.10.** Prove using Barr–Beck–Lurie that the category  $\operatorname{CAlg}_k$  of commutative k-algebras is monadic over  $\operatorname{Mod}_k$  and describe the monad (meaning its structure as an algebra object in the monoidal category of endofunctors of  $\operatorname{Mod}_k$ ).

## References

- Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, Hodge cycles, motives, and Shimura varieties., Springer-Verlag, Berlin-New York, 1982. MR 654325
- [2] W. G. Dwyer and J. Spaliński, Homotopy theories and model categories, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126. MR 1361887