DAG Exercises - 3 - Flavors of derived commutative rings

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III.1. Show that the Day convolution symmetric monoidal structure on $\mathrm{GrMod}_k^{\otimes_\mathrm{D}}$ described in class is indeed a symmetric monoidal category. Precisely, it should satisfy the axioms of a tensor category as layed out in [1, Sec. II.1]. Show the same thing for the Koszul symmetric monoidal structure $\mathrm{GrMod}_k^{\otimes_\mathrm{K}}$.

III.2. Show that $\mathbf{D}^\vee \simeq \mathbf{Z} \oplus \mathbf{Z}(-1)$ admits a unique commutative algebra structure in either $\mathrm{GrMod}_k^{\otimes_\mathrm{D}}$ or $\mathrm{GrMod}_k^{\otimes_\mathrm{K}}$.

III.3. Prove that $\mathbf{D}^\vee$ admits no bicommutative bialgebra structure in $\mathrm{GrMod}_\mathbf{Z}^{\otimes_\mathrm{D}}$. Prove that it admits a unique such structure in $\mathrm{GrMod}_\mathbf{Z}^{\otimes_\mathrm{K}}$. (Hint: don't forget the counit condition!)

III.4. Prove that if $\mathcal{C}$ is a symmetric monoidal category and $B \in \mathrm{cCAlg}(\mathrm{CAlg}(\mathcal{C}))$ is a bicommutative bialgebra, then there is a canonical symmetric monoidal structure on $\mathrm{Mod}_B(\mathcal{C})$ such that the forgetful functor $\mathrm{Mod}_B(\mathcal{C}) \to \mathcal{C}$ is symmetric monoidal.

III.5. Question (I really do not know the answer at the moment): is there a simplicial enrichment for the category of non-negative cdgas over $\mathbf{Q}$ making it into a simplicial model category. Hint: you could attempt to transport the structure using the free functor from non-negative chain complexes and the Dold–Kan correspondence.

III.6. Let $\mathbf{Q}[x_{2n}]$ be the free cdga on a degree $2n$ generator for $n \geqslant 0$. Compute $\pi(\mathbf{Q}[x_{2n}], X)$, the set of homotopy classes of cdga maps $\mathbf{Q}[x_{2n}] \to X$, as a functor in X. (The same argument works for free cdgas on odd generators.)

III.7. Compute a factorization of $\mathbf{Q}[x_0] \to \mathbf{Q}[x_0] \times \mathbf{Q}[x_0]$ as $\mathbf{Q}[x_0] \to \mathbf{Q}[x_0]^I \to \mathbf{Q}[x_0] \times \mathbf{Q}[x_0]$ where the first map is a trivial cofibration and the second map is a fibration. (Thus, $\mathbf{Q}[x_0]^I$ is a path object for $\mathbf{Q}[x_0]$.)

III.8. Consider $\mathbf{Q}[x_{2n}]/(x_{2n}^2)$ as a cdga with 0 differential where x_{2n} has degree $2n$. On the other hand, consider the cdga A defined as the pushout

$$\begin{array}{ccc} \mathbf{Q}[x_{4n}] & \xrightarrow{x_{4n} \mapsto x_{2n}^2} & \mathbf{Q}[x_{2n}] \\ \downarrow & & \downarrow \\ \mathbf{Q}[x_{4n}] \otimes \Lambda_\mathbf{Q}(x_{4n+1}) & \longrightarrow & A, \end{array}$$

where in the bottom left $d(x_{4n+1}) = x_{4n}$. Of course, A and $\mathbf{Q}[x_{2n}]/(x_{2n}^2)$ are weakly equivalent. (Construct a specific quasi-isomorphism!). Complete the following exercises, referring to [2, Sec. 4] for further model categorical details.

(a) Prove that the cdga A is fibrant and cofibrant in the model category structure on non-negative cdgas over $\mathbf{Q}$.
(b) Prove that the cdga $\mathbf{Q}[x_{2n}]/(x_{2n}^2)$ is fibrant but not cofibrant.
(c) Compute $\pi(A, X)$, the set of maps $A \to X$ up to homotopy, as a functor of the cdga X.
(d) Prove that $\pi^r(\mathbf{Q}[x_{2n}]/(x_{2n}^2), X)$ cannot agree with $\pi(A, X)$ by finding an appropriate choice of X.

III.9. Is $\mathbf{Q}[x_0]$ cofibrant when viewed as a cdga? What about $\mathbf{Q}[x_0^{\pm 1}]$?

III.10. Prove using Barr–Beck–Lurie that the category CAlg_k of commutative k-algebras is monadic over Mod_k and describe the monad (meaning its structure as an algebra object in the monoidal category of endofunctors of Mod_k).

References

[1] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties.*, Springer-Verlag, Berlin-New York, 1982. MR 654325

[2] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126. MR 1361887