INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY
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MINICOURSE ON DG CATEGORIES

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Problem sets 4-5
The Yoneda embedding. Pre-triangulated DG categories.

(Problems marked with $*$ are slightly harder, or perhaps they concern some side topics, so I consider them less important.)

Everywhere below, $\mathcal{A}$ is a DG category, and $C(\mathcal{A})^{sf} \subset C(\mathcal{A})$ is the full subcategory of semifree modules.

1. Let $F \in C(\mathcal{A})$ be free. Show that it is homotopically projective: For any acyclic $M \in C(\mathcal{A})$, $\mathrm{Hom}(F, M)$ is acyclic.

2*. More generally, show that semifree modules are acyclic.

3. Suppose $x, y \in \mathcal{A}$. What does it mean that $y \simeq x[1]$? The 'Yoneda-style' definition is that we have an isomorphism $h_y \simeq h_x[1]$ in $C(\mathcal{A}^{op})$. Verify that this is equivalent to the 'naive' definiton: there is a invertible degree one morphism $y \to x$. (Make sure you understand, or even better write down the definition, of any new notions like 'invertible degree one morphism'.)

4. Continuing with the previous problem, can you do the same for cones? That is, suppose $f : x \to y$ is a morphism in $\mathcal{A}$; what do we mean when we say $z \in \mathcal{A}$ is the cone of the morphism? Of course, there is the Yoneda-style definition, but can you reformulate it directly?

5*. In the previous two questions, we considered the *strict* version of cones and translations. However, the homotopy (or perhaps 'quasi'?) version is more useful: instead of asking for an isomorphism in the category $C(\mathcal{A}^{op})$, we can require only a quasi-isomorphism. What does the direct version look like in this case?