INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY
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MINICOURSE ON DG CATEGORIES

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Problem sets 6-7
Compactly generated DG categories. Quasicoherent sheaves.

As before, $\mathcal{A}$ is an arbitrary DG category.

1. Show that the image of the Yoneda embedding $\mathcal{A} \to \mathrm{Ind}(\mathcal{A}) = D(\mathcal{A}^{op})$ consists of compact objects.

2*. Show that if $M \in D(\mathcal{A})$ is compact, it is perfect. (For a simpler problem, assume that M is the cone of a map between two free modules, possibly of infinite rank.)

3*. Put $R = \Bbbk[x]$ be the classical polynomial algebra and consider the category $C(R)$ (rather than $D(R)$). It is a cocomplete DG category. Which of the following objects compactly generate the category:

(1) M where M is a finitely generated R-module;
(2) $(R/f^d R)$, where f is either 0 or irreducible and $d > 0$;
(3) R/fR, where f is either 0 or irreducible;
(4) R?

4. Let $R = \Bbbk[x]$ be the classical polynomial algebra, and let M be $\Bbbk$ considered as an R-module with the trivial action of x.

(1) Verify that $M \in D(R)$ is perfect and compute the DG ring of endomorphisms $\mathrm{End}_R(M)$.
(2) Describe the full subcategory compactly generated by M as explicitly as possible. What condition does $N \in D(R)$ need to satisfy to belong to this category?
(3) Describe the right orthogonal complement to M in $D(R)$ as explicitly as possible.

(Remark: The subcategory generated by M is equivalent to $D(\mathrm{End}_R(M)^{op})$; this is a version of the Koszul transform.)

5. Let C be a smooth projective connected curve over an algebraically closed field $\Bbbk$. Let $x \in X$ be a point. Show that the invertible sheaves $\mathcal{O}_C$ and $\mathcal{O}_C(-x)$ compactly generate the category $\mathrm{QCoh}(X)$.

6. Show that the category $\mathrm{QCoh}(\mathbb{P}^n)$ is compactly generated by invertible sheaves $\mathcal{O}, \dots, \mathcal{O}(n)$. Conclude that the category is (quasi)equivalent to the derived category of a certain classical ring.

(One way to approach this problem, due to A. Beilinson, is to use the Koszul resolution of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$. I suggest you try to prove this by induction instead.)

7*. Let $f : Y \to X$ be a locally closed embedding (or, more generally, any quasi-affine morphism.) Show that if $\mathcal{A} \subset \mathrm{QCoh}(X)$ compactly generates $\mathrm{QCoh}(X)$, then $f^*(\mathcal{A})$ compactly generates $\mathrm{QCoh}(Y)$. Here and everywhere in this course, all functors are derived by default. Can you formulate a categorical statement generalizing this problem?

(The last two problems imply that the category $\mathrm{QCoh}(X)$ for a quasi-projective X is compactly generated without using the Thomason-Trobaugh Theorem.)