INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY
MSRI, SUMMER 2023
MINICOURSE ON DG CATEGORIES

DIMA ARINKIN

Problem sets 8-9
Operations on DG categories. The Fourier-Mukai transform.

1. (Problem from the lecture) Let

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

be a short exact sequence of cocomplete DG categories. Suppose that we have a collection of $\{F_\alpha\} \subset \mathcal{B}^c$ whose images (compactly) generate $\mathcal{C}$ and a collection $\{G_\beta\} \subset \mathcal{A}^c \subset \mathcal{B}^c$ of objects that generate $\mathcal{A}$. Show that the union of the two collections

$$\{F_\alpha\} \cup \{G_\beta\}$$

generates $\mathcal{B}$. (In particular, this proves $\mathcal{B}$ is compactly generated.)

2*. (A little more on the bar resolution) Let R be a ring, and let $B_\bullet$ be a resolution of R by free R-bimodules. Recall that we can associate to any $M^\bullet \in K(R)$ the total complex of the bicomplex

$$B_\bullet \otimes M^\bullet;$$

this provides a functor from the category $K(R)$ to its full subcategory of semifree modules.

Show that the functor is the right adjoint of the inclusion functor.

3. Let $R = \bigoplus_{i \geq 0} R_i$ be a non-negatively graded ring. Consider the derived category $D(R)^{gr}$ of *graded* R-modules. (Hopefully, it is clear what this means: the grading of modules is independent of the cohomological grading.) Prove that the category is compactly generated and find a set of generators.

4. Continuing with the previous problem, prove that the category $D(R)^{gr}$ is not equivalent to the category $D(S)$ for any dg ring S.

5*. Consider the category $D(\Bbbk[x_0, \dots, x_n])^{gr}$, and let

$$D(\Bbbk[x_0, \dots, x_n])^{gr}_0 \subset D(\Bbbk[x_0, \dots, x_n])^{gr}$$

be the full subcategory of complexes on whose cohomology all x_i act locally nilpotently. Show that the subcategory is compactly generated, and identify the quotient

$$D(\Bbbk[x_0, \dots, x_n])^{gr} / D(\Bbbk[x_0, \dots, x_n])^{gr}_0$$

with $\mathrm{QCoh}(\mathbb{P}^n)$.

6*. In the notation of the previous problem, consider the colocalization functor

$$D(\Bbbk[x_0, \dots, x_n])^{gr} \to D(\Bbbk[x_0, \dots, x_n])^{gr}_0.$$

Show that the functor is fully faithful on the subcategory of compact objects. This implies that the 'larger' category $D(\Bbbk[x_0, \dots, x_n])^{gr}$ can be obtained from the 'smaller' category $D(\Bbbk[x_0, \dots, x_n])^{gr}_0$ by 'renormalization': taking the ind-completion of its full subcategory. (It is called 'renormalization' because we change which objects are considered compact.)