DAG Exercises - 5 - The cotangent complex

Benjamin Antieau

Summer 2023

V.1. Prove that if R and S are in CAlg_k and if $R \otimes_k S$ denotes, for the moment, the classical tensor product, then $\Omega^1_{R \otimes_k S/k} \simeq (\Omega^1_{R/k} \otimes_k S) \oplus (R \otimes_k \Omega^1_{S/k})$, where all tensor products are classical.

V.2. Prove that $\Omega^1_{R/k} \cong I/I^2$ where I is the kernel of the multiplication map $R \otimes_k R \to R$.

V.3. Prove that if $k \to R \to S$ are maps of connective derived commutative rings, then there is a natural cofiber sequence

$$S \otimes_R \mathcal{L}_{R/k} \to \mathcal{L}_{S/k} \to \mathcal{L}_{S/R}.$$

V.4. Show that if G is a group scheme, then the cotangent complex of Spec $k \to G$ corresponding to the identity element of G is equivalent to \mathfrak{g}^{\vee} , the k-linear dual of the Lie algebra of G.

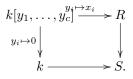
V.5. Show that if $R, S \in \text{DAlg}_k^{\text{cn}}$, then $L_{R \otimes_k S/k} \simeq (L_{R/k} \otimes_k S) \oplus (R \otimes_k L_{S/k})$.

V.6. Let k be a commutative ring in which 2 is invertible. Let $R = k[x, y]/(x^2 + y^2 - 1)$ and let $X = \operatorname{Spec} R$ as a derived stack. Let $S \in \operatorname{DAlg}_k^{\operatorname{cn}}$. Compute $\pi_i X(S)$ (with any given base point of X(S)) in terms of $\pi_i S$. Attempt to do this two ways, first by writing R as a pushout, second by computing and using the cotangent complex of R.

V.7. Let $R = \operatorname{LSym}_k M$ for $M \in D(k)_{\geq 0}$. Compute $L_{R/k}$.

V.8. Let $R = k[x]/(x^2)$. Compute $L_{R/k}$ and $L_{k/R}$. Why doesn't your solution contradict Avramov's theorem?

V.9. Suppose that $R \in \text{CAlg}_k$ and x_1, \ldots, x_c is any set of elements in R. Define $S = R//(x_1, \ldots, x_c)$ as the pushout



When is S discrete? Compute $L_{S/R}$.

V.10. Suppose that $k \to R \to S$ be maps in $\text{DAlg}_{\mathbf{Z}}^{\text{cn}}$. The associated Kodaira–Spencer class $\text{KS}_{S/R}$, which despite the notation depends on k, is the map $L_{S/R} \to (S \otimes_R L_{R/k})[1]$ associated to the boundary map in the conormal sequence. It is typically viewed as an element of $\text{Ext}_S^1(L_{S/R}, S \otimes_R L_{R/k})$. Now, suppose that \widetilde{R} is a square-zero extension of R by $M \in D(R)_{\geq 0}$, classified by a map d: $L_{R/k} \to M[1]$. Show that the obstruction to finding a deformation of S to \widetilde{R} is a class in $\text{Ext}_S^2(L_{S/R}, S \otimes_R M)$ given as the composition of $\text{KS}_{S/R}$ and $S \otimes_R d: S \otimes_R L_{R/k} \to S \otimes_R M[1]$. Show that the space of such deformations has π_0 given by $\text{Ext}_S^1(L_{S/R}, S \otimes_R M)$.