DAG Exercises - 5 - The cotangent complex

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V.1. Prove that if R and S are in CAlg_k and if $R \otimes_k S$ denotes, for the moment, the classical tensor product, then $\Omega^1_{R\otimes_k S/k} \simeq (\Omega^1_{R/k} \otimes_k S) \oplus (R \otimes_k \Omega^1_{S/k})$, where all tensor products are classical.

V.2. Prove that $\Omega^1_{R/k} \cong I/I^2$ where I is the kernel of the multiplication map $R \otimes_k R \to R$.

V.3. Prove that if $k \to R \to S$ are maps of connective derived commutative rings, then there is a natural cofiber sequence

$$S \otimes_R \mathrm{L}_{R/k} \to \mathrm{L}_{S/k} \to \mathrm{L}_{S/R}.$$

V.4. Show that if G is a group scheme, then the cotangent complex of $\operatorname{Spec} k \to G$ corresponding to the identity element of G is equivalent to $\mathfrak{g}^\vee$, the k-linear dual of the Lie algebra of G.

V.5. Show that if $R, S \in \mathrm{DAlg}_k^{\mathrm{cn}}$, then $\mathrm{L}_{R\otimes_k S/k} \simeq (\mathrm{L}_{R/k} \otimes_k S) \oplus (R \otimes_k \mathrm{L}_{S/k})$.

V.6. Let k be a commutative ring in which 2 is invertible. Let $R = k[x, y]/(x^2 + y^2 - 1)$ and let $X = \operatorname{Spec} R$ as a derived stack. Let $S \in \mathrm{DAlg}_k^{\mathrm{cn}}$. Compute $\pi_i X(S)$ (with any given base point of $X(S)$) in terms of $\pi_i S$. Attempt to do this two ways, first by writing R as a pushout, second by computing and using the cotangent complex of R.

V.7. Let $R = \mathrm{LSym}_k M$ for $M \in \mathrm{D}(k)_{\geqslant 0}$. Compute $\mathrm{L}_{R/k}$.

V.8. Let $R = k[x]/(x^2)$. Compute $\mathrm{L}_{R/k}$ and $\mathrm{L}_{k/R}$. Why doesn't your solution contradict Avramov's theorem?

V.9. Suppose that $R \in \mathrm{CAlg}_k$ and $x_1, \ldots, x_c$ is any set of elements in R. Define $S = R//(x_1, \ldots, x_c)$ as the pushout

$$\begin{array}{ccc} k[y_1,\ldots,y_c] & \xrightarrow{y_i \mapsto x_i} & R \\ {\scriptstyle y_i\mapsto 0}\downarrow & & \downarrow \\ k & \longrightarrow & S. \end{array}$$

When is S discrete? Compute $\mathrm{L}_{S/R}$.

V.10. Suppose that $k \to R \to S$ be maps in $\mathrm{DAlg}_{\mathbf{Z}}^{\mathrm{cn}}$. The associated Kodaira–Spencer class $\mathrm{KS}_{S/R}$, which despite the notation depends on k, is the map $\mathrm{L}_{S/R} \to (S \otimes_R \mathrm{L}_{R/k})[1]$ associated to the boundary map in the conormal sequence. It is typically viewed as an element of $\mathrm{Ext}^1_S(\mathrm{L}_{S/R}, S \otimes_R \mathrm{L}_{R/k})$. Now, suppose that $\widetilde{R}$ is a square-zero extension of R by $M \in \mathrm{D}(R)_{\geqslant 0}$, classified by a map $\mathrm{d}\colon \mathrm{L}_{R/k} \to M[1]$. Show that the obstruction to finding a deformation of S to $\widetilde{R}$ is a class in $\mathrm{Ext}^2_S(\mathrm{L}_{S/R}, S \otimes_R M)$ given as the composition of $\mathrm{KS}_{S/R}$ and $S \otimes_R \mathrm{d}\colon S \otimes_R \mathrm{L}_{R/k} \to S \otimes_R M[1]$. Show that the space of such deformations has π_0 given by $\mathrm{Ext}^1_S(\mathrm{L}_{S/R}, S \otimes_R M)$.