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Holomorphic Differentials in Mathematics and Physics

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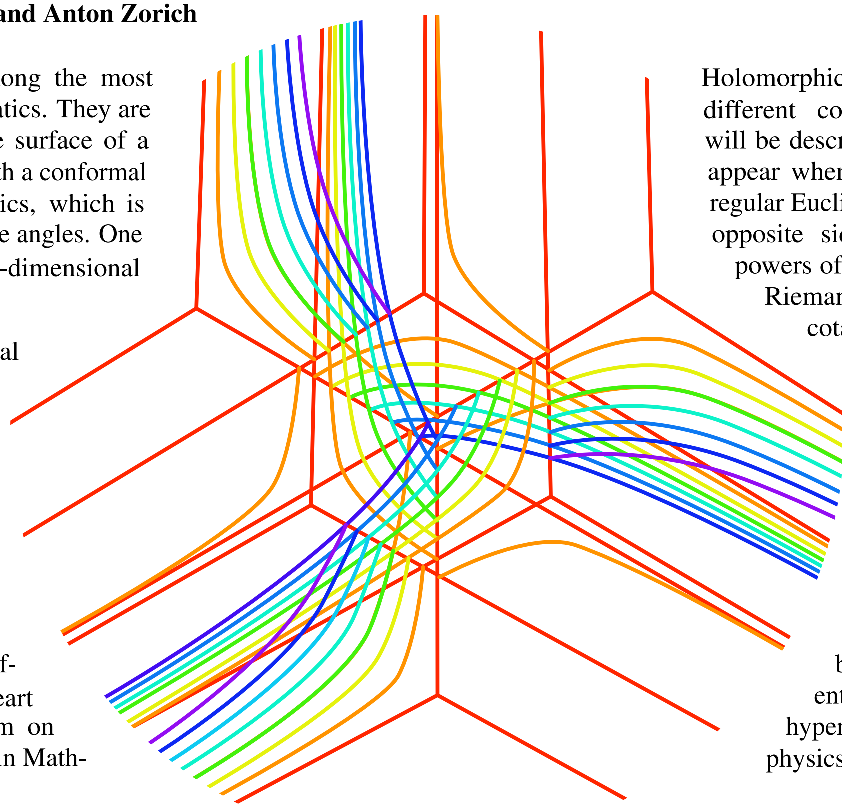
Riemann surfaces are among the most studied objects in mathematics. They are compact surfaces (like the surface of a donut) that are endowed with a conformal class of Riemannian metrics, which is essentially a way to measure angles. One can also think of them as 1-dimensional complex manifolds.

They arise in differential geometry, complex analysis, algebraic geometry differential equations, theoretical physics, dynamical systems, and many other fields. Riemann surfaces can be endowed with additional structure, and one such structure — holomorphic differentials — lies at the heart of this semester's program on Holomorphic Differentials in Mathematics and Physics.



Holomorphic differentials arise in many different contexts, several of which will be described below. They naturally appear when gluing a surface out of a regular Euclidean polygon by identifying opposite sides; they are sections of powers of the canonical bundle of the Riemann surface; they can arise as cotangent vectors to the moduli space of Riemann surfaces.

One particularly important occurrence of holomorphic differentials is that they form the base of the Hitchin integrable system, as we will recall below. This is the beginning of a far-reaching connection between holomorphic differentials, representation varieties, hyperKähler geometry, theoretical physics, and more.

(continued on page 10)

A spectral network: This geometric structure on a Riemann surface X arises naturally from consideration of the quantum field theories $S(X, g)$.

Meet Future Mathematicians at the National Math Festival . . . and More

The **National Math Festival** invites children and adults alike to experience the joy, beauty, and relevance of mathematics. On May 4, around 12,000 people in Washington, DC, attended this year's festival, and a further 26,000 joined in associated activities across the country. There's a full report on pages 8–9.

Microlocal analysis is applied to such a vast range of problems that it can sometimes seem more like an organizing principle than a specific area of study — read about **this semester's program on Microlocal Analysis** and its applications on page 4.

And finally, the **Puzzles Column** honors and remembers **Elwyn Berlekamp**. With Joe Buhler, Elwyn wrote the Puzzles Column for nearly 20 years, and for this issue Joe and Tanya Khovanova have collected a special set of problems that evoke Elwyn and his love of puzzles (pages 14–15).

The View from MSRI

David Eisenbud, Director

I'm now back as Director (term ending 2022), after a year during which Hélène Barcelo was Acting Director, with Michael Singer as part-time Associate Director. They did a great job of managing this wonderful place! During this "sabbatical" I worked on raising funds for the many MSRI activities that are not funded by the NSF, and undertook two large projects: writing a strategic vision that was adopted by the Board last fall, and writing the re-competition proposal for the NSF, of which more later. I'm grateful for the engagement and input from many people in these projects, greatly strengthening the result. It warms me to interact with so many people — mathematicians and others — who care deeply about the direction of this institute.

Status of MSRI's NSF Recompetition Proposal

This is a proposal for major support over the next five years — thus of immense importance to MSRI. We started preparing in 2018, started serious writing in January 2019, submitted the proposal in March, got referee reports in the summer, and made a presentation at a "reverse site visit" on September 12.

What is a reverse site visit? Instead of coming to MSRI to meet members and staff and see the facilities for a couple of days, the NSF invited us to come to Washington to make a three-hour presentation. Hélène and I were invited to bring up to six others to help us, and I was delighted with the people, including both distinguished mathematicians and high-placed administrators, who were willing to make the trek to support MSRI.

The next step in this year-long process: Wait! We were told we'd hear something more from the NSF in December.

Statistics are byproducts of the NSF review. For example, over the last 10 years almost 20,000 visitors participated in MSRI's scientific activities, of whom 64% were from US universities, 30% were women, and 13% members of other underrepresented groups. Over 500 institutions were represented at our 32 semester-long programs, and students from over 250 institutions attended our Summer Graduate Schools.

Fundraising

A year ago MSRI hired Annie Averitt as Director of Advancement and External Relations (a.k.a., chief fundraiser) and we subsequently added Lynda Wright as Assistant Director. The increased staffing is beginning to pay off, with major new foundation donors and a new corporate partner, as well as robust development income overall.

Roughly speaking, MSRI's budget now comes half from the NSF and half from other sources. The flexibility granted by private fundraising has allowed us to improve our support of research, provide better support for members coming with families, and undertake some projects for the public understanding of mathematics. We have begun to build up an endowment, which can now contribute about 10% of our overall budget. Building the endowment further so that MSRI will be able to serve the math community in the long run is a high priority.

Celebrating Maryam Mirzakhani

One of the things that private fundraising has allowed us to do is to produce a documentary, *Secrets of the Surface: The Mathematical Vision of Maryam Mirzakhani*. The film premiere will be at the Joint Mathematics Meetings in Denver this winter (5:15 pm on Friday, January 17, 2020 in room 207 of the Convention Center).

Directed by George Csicsery (*N is a Number* and *Navajo Math Circles*), the one-hour film features some extraordinary footage from Iranian classrooms as well as commentary on Maryam's work by many distinguished mathematicians, including a moving piece by Alex Eskin, one of Maryam's closest collaborators.

Breakthrough Prizes

Congratulations to Alex Eskin on the 2019 Breakthrough Prize! It's fun to see how those celebrated at the Breakthrough ceremony this year have passed through MSRI as their careers advanced: Alex himself was a member in 2007 and then Simons Professor in 2015. Tim Austin was a visiting graduate student in 2008 and then a member in 2011. Emmy Murphy attended a workshop in 2018 and was a lecturer in one of our summer graduate schools the same year. Xinwen Zhu was a member in 2014. Congratulations to them all!

Evaluating MSRI

How expensive would it be to get a truly arm's-length evaluation of MSRI? We got one for free, as another byproduct of the NSF review! Here are a some of the comments from the anonymous reviewers of our NSF proposal who were chosen by the NSF:

> “ MSRI has, over 35 years, become one of the premier mathematics research institutes in the world. The list of past postdocs and other luminaries speaks volumes as does the administration's ability to point to a host of fundamental developments that were sparked through their programs.

> “ MSRI has been a tremendous and impactful national resource for research mathematicians for decades.

> “ MSRI has moved beyond providing opportunities to members of underrepresented groups and progressed in some cases to removing obstacles, as seen for example in providing full funding for additional childcare related expenses incurred by women program members and workshop participants . . .

> “ A forward-thinking proposal from a successful institute; particularly notable is MSRI's success in leveraging NSF money with donations and other grants enabling them to improve their facilities and offer wider ranging programs.

> “ The proposal is extremely strong in mathematical content, variety of topics and programs, broad participation and impact, richness of opportunities and connections.

> “ This proposal is superior in all possible ways.

Public Understanding of Mathematics — Highlights

AMS/MSRI Congressional Briefing Report

MSRI and the American Mathematical Society (AMS) host two congressional briefings on mathematical topics each year in Washington, DC, to inform members of Congress and congressional staff about new developments made possible through federal support of basic science research.

On June 13, 2019, **Jon Kleinberg**, Tisch University Professor in the departments of computer science and information science at Cornell University, spoke on Capitol Hill on "Addressing Threats and Vulnerabilities in Critical Interconnected Systems." A vital feature of many critical systems in society is their connectivity — they are built from large numbers of components linked together in a network. This structure makes it possible to build them at large scales, but it also puts them at risk of cascading breakdowns, when a problem in one component spreads to others.

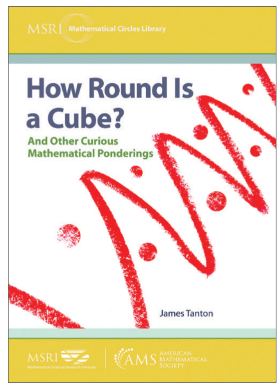

Jon Kleinberg

Through mathematical models originally developed for epidemic diseases, where a small change in the connectivity of the population or the infectiousness of the disease can lead to large changes in the reach of the outbreak, we can consider how these models apply when developing detection techniques and countermeasures for risks to highly interconnected systems, including malware on the Internet and cascading failures in banking systems. A full report of Kleinberg's talk can be read in the October 2019 issue of the AMS Notices.

The next congressional briefing will take place in December 2019, featuring current AMS President **Jill Pipher** (Brown University) speaking on cryptography. You can learn more and view short films about past events: msri.org/congress.

Mathematical Circles Library: New Volume by James Tanton

Volume 23 in the Mathematical Circles Library is here! *How Round is a Cube? And Other Curious Mathematical Ponderings*, by **James Tanton** (Mathematical Association of America), is a collection of 34 curiosities, each a quirky and delightful gem of mathematics and each a shining example of the joy and surprise that mathematics can bring.



Intended for the general math enthusiast, each essay begins with an intriguing puzzle, which either springboards into or unravels to become a wondrous piece of thinking. The essays are self-contained and rely only on tools from high-school mathematics (with only a few pieces that ever-so-briefly brush up against high-school calculus).

MSRI and the AMS publish this book series as a service to young people, their parents and teachers, and the mathematics profession. Explore the entire collection at bookstore.ams.org/MCL.

MSRI's New Ombuds

Catherine Glaze joined MSRI as its inaugural Ombuds in May 2019. Catherine is a Stanford graduate (B.A. 1980, J.D. 1985), who joined Stanford's staff in 2000 and served for more than fifteen years as the Associate Dean for Student Affairs in the Law School before becoming the University's Title IX Coordinator in October 2015 until her retirement in August 2018. She previously was an attorney in private practice and held instructional and administration roles at Golden Gate University School of Law.


Catherine Glaze

As Title IX Coordinator, Catherine oversaw the investigation and resolution of all student-related matters involving sexual assault, sexual misconduct, sexual harassment, stalking and relationship violence. As Dean of Students, she counseled students and served as the primary liaison between students and faculty in addition to developing and implementing academic policies and procedures and serving as a member of the Law School's senior management team.

Catherine served the university through a number of other roles as well, including as chair of the Board on Judicial Affairs, which oversees campus policies on student conduct; as a sexual harassment adviser in the Law School; as a member of the Grievance Advisory Board that hears employee grievances; and, as a member of the search committee for the Vice Provost of Student Affairs.

Contents

Microlocal Analysis and its Applications

Steve Zelditch

Microlocal analysis is analysis in phase space. Phase space is a symplectic manifold M, most often the cotangent bundle T^*X of a manifold, or a Kähler manifold (M, ω). The geometry of phase space is symplectic geometry and dynamics of Hamiltonian flows. Microlocal analysis begins with the quantization of symplectic geometry, in particular the quantization of real-valued functions $a \in C^\infty(M)$ on M as self-adjoint operators $Op(a)$, and symplectic diffeomorphisms $\chi : M \to M$ as unitary operators U_χ on a Hilbert space. The basic ideas of quantizing functions as self-adjoint operators and symplectic transformations as unitary operators can be found in classic books of P. Dirac and H. Weyl in the 1920s. One might think of microlocal analysis as the application of quantum mechanical ideas to a vast array of problems in geometry and mathematical physics, often with tenuous connections to quantum mechanics.

Microlocal means local in phase space, for example, local in T^*X rather than simply local in X. Symplectic diffeomorphisms can be used to put equations into local normal forms. Quantizations of symplectic diffeomorphisms can be used to put partial differential operators into microlocal normal forms. Although microlocal suggests "very local," one of its fundamental strengths is that it is also global: it allows analysts to exploit global geometry and dynamics to study solutions of partial differential equations.

In current research (and in the MSRI program), microlocal analysis is applied to such a vast range of problems that it seems more like an organizing principle than a specific area of study: microlocal analysis is applied (i) to spectral or scattering theory of Laplacians (and more general Schrödinger operators), for example, to inverse problems for Laplacians or to quantum chaos; (ii) to dynamical problems such as resonances of Anosov flows; (iii) to analysis on singular spaces; and (iv) to nonlinear stability problems in general relativity.

We begin with the setting of linear and quadratic Hamiltonians on $\mathbb{R}^n$; it is the "linear algebra," which is generalized to "calculus on manifolds" by microlocalization.

Canonical Quantization: Heisenberg and Metaplectic Groups

Canonical quantization means the Schrödinger representation of the Heisenberg algebra and group (explaining why it is canonical). The algebra is represented as follows: Let $X = (x_1, \ldots, x_n)$ and $q \cdot X = \sum_j q_j x_j$. They act on $L^2(\mathbb{R}^n)$ as multiplication operators, $q \cdot X f(x) = \langle q, x \rangle f(x)$. Let $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ and $p \cdot P = \frac{1}{i}\sum_j p_j \frac{\partial}{\partial x_j}$. These operators satisfy $[X_j, D_k] = i\delta_{jk}$, the Heisenberg commutation relations.

Weyl defined the corresponding Heisenberg group representation, defined by

$$U(\alpha) = e^{\frac{2\pi i}{\hbar}\alpha P}, \quad V(\beta) = e^{\frac{2\pi i}{\hbar}\beta Q}.$$

These unitary operators act by

$$U(\alpha) f(q) = f(q + \alpha), \quad V(\beta) f(q) = e^{\frac{2\pi i}{\hbar}} f(q)$$

and satisfy the Weyl commutation relations,

$$S(\alpha, \beta) = e^{-\frac{i}{2}\langle \alpha, \beta \rangle} U(\alpha) V(\beta) = e^{\frac{i}{2}\langle \alpha, \beta \rangle} V(\beta) U(\alpha).$$

The group algebra generated by these operators (the integrated representation) consists of the Weyl pseudo-differential operators,

$$Op_\hbar^w(a) u(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a_\hbar\left(\frac{1}{2}(x+y), \xi\right) e^{\frac{i}{\hbar}\langle x-y, \xi \rangle} u(y)\, dy\, d\xi.$$

The (Weyl) symbol is a function on $T^*\mathbb{R}^n$. If one chooses it to be (approximately) the characteristic function $\mathbf{1}_{E \times F}$ of a phase space box $E \times F \subset T^*\mathbb{R}^n$, then $Op_h(\mathbf{1}_{E \times F})u$ "microlocalizes" u to $E \times F$, that is, $Op_h(\mathbf{1}_{E \times F})u(x)$ is small outside of E and its Fourier transform is small outside of F.

Quadratic Hamiltonians: The symplectic group and its metaplectic representation. The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ acts by conjugation on the Weyl–Heisenberg representation, rotating the pair $(Q_j, P_j) : \mathcal{F} Q_j \mathcal{F}^* = P_j, \mathcal{F} P_j \mathcal{F}^* = -Q_j$. In fact, $\mathcal{F}$ belongs to a unitary representation of the symplectic group $Sp(n, \mathbb{R})$, or more precisely its double cover $Mp(n, \mathbb{R})$, the metaplectic group.

A symplectic matrix has block form, $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The metaplectic representation has the form

$$\mu \begin{pmatrix} A & B \\ C & D \end{pmatrix} f(x) = i^{n/2} (\det B)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{2\pi i S(x,y)} f(y)\, dy,$$

with

$$S(x,y) = -\frac{1}{2}\left(x D B^{-1} x + y B^{-1} x - \frac{1}{2} y B^{-1} A y \right).$$

One of the cornerstone results of Fourier integral operator theory is the Egorov theorem on conjugations of pseudo-differential operators by Fourier integral operators quantizing symplectic diffeomorphisms. The simplest case is to conjugate a Weyl pseudo-differential operator $Op_h^w(a)$ on $L^2(\mathbb{R}^n)$ by a metaplectic operator $\mu(\mathcal{A})$. One then has the exact formula,

$$\mu(\mathcal{A})^* Op_h^w(a) \mu(\mathcal{A}) = Op_h^w(a \circ \mathcal{A}),$$

where $\mathcal{A} \circ a(x, \xi) = a(\mathcal{A}(x, \xi))$.

This formula, and its generalization, are the basis for semiclassical analysis, the analysis of the semiclassical limit $\hbar \to 0$ whereby quantum mechanical objects (such as $\mu(\mathcal{A}), Op_h^w(a)$) tend to classical ones (such as $\mathcal{A}, a$).

Weyl symbols and Wigner distributions. In 1932, Wigner considered the question: Can one construct from a wave function $f \in L^2(\mathbb{R}^n)$ a phase space density? He answered the question by constructing what are now called Wigner distributions. Given two normalized functions, $f, g \in L^2(\mathbb{R}^d)$, the semiclassical Wigner distribution $W_{f,g;\hbar}(x,\xi)$ is the function on $T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$ defined by

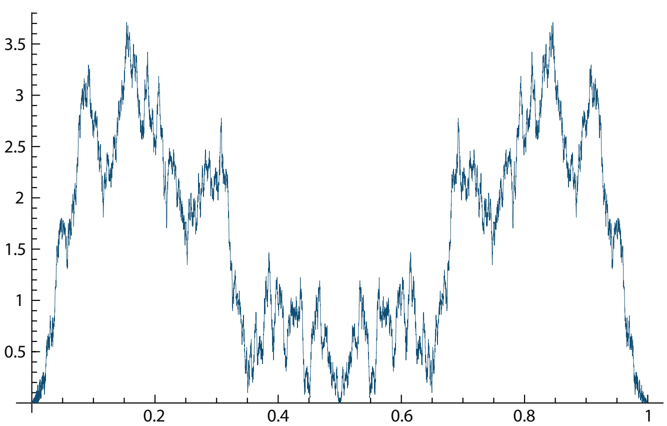
$$W_{f,g;\hbar}(x,\xi) := \int_{\mathbb{R}^d} f(x+\frac{v}{2})\overline{g(x-\frac{v}{2})}e^{-\frac{i}{\hbar}v\xi}\frac{dv}{(2\pi\hbar)^d}.$$

Here, $\hbar$ is Planck's constant, treated as a semiclassical parameter in the sense that we are interested in relations to classical mechanics as $\hbar \to 0$. In particular, when $f = g$, Wigner regarded $W_{f,f;\hbar}(x,\xi)$ as the phase space density associated to f. Roughly speaking, $W_{f,f;\hbar}(x,\xi)dxd\xi$ is supposed to be the probability density of finding the particle at the phase space point (x,ξ), that is, to have position x and momentum ξ.

Some of the key objects of study in quantum mechanics are matrix elements $\langle Op_\hbar^w(a)\phi_j, \phi_k\rangle$ of pseudo-differential operators in an orthonormal basis $\{\phi_j\}$, usually an orthonormal basis of eigenfunctions on an interested quantum Hamiltonian. A key formula is that

$$\langle Op^w(a)f, f\rangle = \int_{T^*\mathbb{R}^d} a(x,\xi)W_{f,f}(x,\xi)dxd\xi,$$

representing matrix elements by concrete integral formula.



This image is the graph of the primitive $T(b)$ of the boundary value $dT(b)$ of a Laplace eigenfunction on $G/PSL(2,\mathbb{Z})$ on the unit circle, the ideal boundary of the Poincaré disk. (Figure by Stephen D. Miller).

Quantum Chaos on Hyperbolic Surfaces

We now turn to quantum chaos on hyperbolic surfaces $\mathbf{X}_\Gamma = \Gamma\backslash\mathbf{D} = \Gamma\backslash G/K$, a model example where the quantum mechanics is as far from solvable as possible. Here, $G = PSL(2,\mathbb{R}) \simeq PSU(1,1)$, $K = SO(2)$ and $\Gamma \subset G$ is a discrete co-compact subgroup, and $\mathbf{D}$ is the hyperbolic disc.

Quantum chaos is in part concerned with the limits of matrix elements $\langle Op(a)u_{ir_j}, u_{ir_j}\rangle$ of zeroth-order pseudo-differential operators with respect to an orthonormal basis $\{u_{ir_j}\}_{j=0,1,2,\dots}$ of real-valued eigenfunctions:

$$\triangle u_{ir_j} = -\lambda_j u_{ir_j},$$

where Δ is the Laplacian of the hyperbolic metric and where $\lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \dots$ denotes the spectrum of the Laplacian on $\mathbf{X}_\Gamma$, repeated according to multiplicity.

Morally, $\langle Op(a)u_{ir_j}, u_{ir_j}\rangle = \int_{S^*\mathbf{X}_\Gamma} a dW_{r_j}$, where W_j is the "Wigner distribution" in this setting. Henceforth we write u_h for u_{ir_j}, where $h = h_j = r_j^{-1}$. A standard result is that $\langle Op(a)u_{h_j}, u_{h_j}\rangle \to \int_{S^*\mathbf{X}_\Gamma} a d\mu_L$, where $d\mu_L$ is Liouville measure, for "almost all" h_j. A natural question is whether there exist exceptional sparse subsequences, and what constraints can be put on the limits of their Wigner distributions (known as quantum limits).

Fractal uncertainty principle. A recent breakthrough on this sparse subsequence problem for hyperbolic surfaces is the following result due to Dyatlov–Jin (generalized to any negatively curved surface by Dyatlov–Jin–Nonnenmacher). We use semiclassical notation where $h_j = \lambda_j^{-1}$ and denote eigenfunctions by u_{h_j} or more simply u_h.

> **Theorem**. Let (M, g) be a compact hyperbolic surface. Let $a \in C_0^\infty(T^*M)$ with $a|_{S^*M}$ not identically zero. Let u_h be an eigenfunction of eigenvalue h^{-2} and $\|u_h\|_{L^2} = 1$. Then there exists a constant C_a independent of h so that, for $h \leqslant h_0(a)$,
>
> $$\|Op_h(a)u_h\|_{L^2} \geqslant C_a.$$

Here, $Op_h(a)$ is the semiclassical pseudo-differential operator with symbol a. If $a(x,\xi) = V(x)$ is a multiplication operator, one gets that $\int_B |u_h|^2 dV \geqslant C_B > 0$, that is, a uniform lower bound (in h) of the L^2 mass on all balls.

> **Corollary**. All quantum limits of sequences of eigenfunctions on compact hyperbolic surfaces have full support in S^*M, that is, charge every open set.

One of the main ingredients in the proof of the theorem above is the so-called FUP (fractal uncertainty principle). It is related to a classical problem of Landau–Slepian–Pollak related to the uncertainty principle of quantum mechanics: Can there exist a function f which is concentrated on an interval A, such that its Fourier transform $\mathcal{F}f$ is concentrated on an interval B? To make this precise, let $P_A = \mathbf{1}_A$ and $Q_B = \mathcal{F}^*\mathbf{1}_B\mathcal{F}$. If there exists f which is ϵ-concentrated on A in the sense that $\int_{X\setminus A}|f|^2 \leqslant \epsilon^2\int_X |f|^2$, and such that $\mathcal{F}f$ is δ-concentrated on B, then $1-\epsilon-\delta \leqslant \|P_A Q_B\|$.

The FUP in the sense of Bourgain, Dyatlov, Jin, Zahl is a kind of generalization where one replaces intervals by regular porous fractal sets. Given $\nu \in (0,1)$ and $0 < \alpha_0 < \alpha_1$, say that $\Omega \subset \mathbb{R}$ is ν-porous on scales α_0 to α_1 if for each interval I of size $|I| \in [\alpha_0, \alpha_1]$ there exists a subinterval $J \subset I$ with $|J| = \nu|I|$ such that $J \cap \Omega = \emptyset$. Given a set X, let $X(s) = X + [-s, s]$ be its s-thickening. There is an additional regularity condition called "Ahlfors–David" regularity, which we won't define. The FUP for δ-regular sets states the following:

> **Proposition**. Let $B(h)$ be a semiclassical Fourier integral operator on $L^2(\mathbb{R})$, of the form
>
> $$Bf(x) = h^{-\frac{1}{2}}\int_{\mathbb{R}} e^{i\Phi(x,y)/h}b(x,y)f(y)dy,$$
>
> with $b \in C_0^\infty(U)$ and $\partial_{xy}^2\Phi \neq 0$ on U. Suppose that $X, Y \subset \mathbb{R}$ are Ahlfors–David δ-regular. Then there exists $\beta > 0$ so that
>
> $$\|\mathbf{1}_{X(h)}B(h)\mathbf{1}_{Y(h)}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant Ch^\beta.$$

Roughly speaking, the FUP is used in the following way. Suppose that $\Omega \subset M$ is an open ball in which $\|\mathbf{1}_\Omega u_h\|_{L^2}$ is "small," that is, in which u_h^2 does not become uniformly distributed. One can propagate the inverse image $\pi^{-1}(\Omega)$ of this set in S^*M under the natural projection to obtain an invariant phase space set in which u_h is microlocally small. Actually, a fundamental obstruction is that one can only relate classical and quantum dynamics on times $|t| \leqslant |\log h|$ less than the "Ehrenfest time." Forward/backward propagation up to the Ehrenfest time produces sets $\Gamma_\pm(|\log h|)$ on whose intersection u_h is "micro-localized" (that is, on the complement of which it is small), namely, $\Gamma_\pm(T)$ is the complement of the flowout of $\pi^{-1}(\Omega)$ under the geodesic flow for $0 \leqslant \mp t \leqslant T$. This complement is a regular porous set and is very "sparse."

Dyatlov–Jin construct rather exotic semiclassical pseudo-differential operators $A_\pm$ which localize to $\Gamma_\pm(|\log h|)$. Hence $A_+A_-u \simeq u$ in the sense that microlocal mass of u in the complement of $\Gamma_+(|\log h|) \cap \Gamma_-(|\log h|)$ is small. Unfortunately, the calculi to which $A_\pm$ belong are incompatible and the product A_+A_- is not a pseudo-differential operator. At this point, the FUP comes to the rescue: It shows that $\|A_+A_-\|_{L^2(M)\to L^2(M)} \leqslant Ch^\beta$ for some $\beta > 0$. Thus, the microlocal mass of u cannot concentrate in $\Gamma_+(|\log h|) \cap \Gamma_-(|\log h|)$, yet it is very small outside this intersection. The contradiction shows that there could not exist a "hole" in the support of the microlocal defect measure of the sequence $\{u_h\}$.

This outline hides a key problem: the FUP is a statement about operators on $\mathbb{R}$, while A_+A_- lives on a hyperbolic surface. Converting the hyperbolic setting to the Fourier transform setting requires Fourier integral operator conjugations of the type mentioned in the previous section, but which of course are very complicated in the curved setting.

Inverse Problems

One of the classical problems in spectral theory of Laplacians is the Kac problem, "Can you hear the shape of the drum?" That is, can you determine a bounded plane domain from its Dirichlet (or Neumann) spectrum. We will assume here that drums are convex and smooth, so that the problem is almost entirely open. Kac himself proved that circular drums are "spectrally determined," that is, are the unique domains up to isometry with their Dirichlet (or, Neumann) spectrum. Until recently, no other bounded smooth plane domain was known to be spectrally determined. In recent work, Hamid Hezari and the author proved that ellipses of small eccentricity are also spectrally determined; the complete list of bounded smooth domains for which the Kac problem has been solved now consists exactly of these ellipses of small eccentricity.

> **Theorem**. There exists $\varepsilon_0 > 0$ such that any ellipse with eccentricity less than ε_0 is uniquely determined by its Dirichlet (or Neumann) Laplace spectrum, among all smooth domains.

Henceforth, we use the term "nearly circular ellipse" as short for "eccentricity less than ε_0."

Key inputs into the proof are the recent dynamical inverse results of Avila–De Simoi–Kaloshin and Kaloshin–Sorrentino. They prove that if a strictly convex smooth plane domain is sufficiently close to an ellipse and is rationally integrable, then it must be an ellipse. Rational integrability means that for every integer $q \geqslant 3$, there is a "convex caustic" of rotation number $\frac{1}{q}$ consisting of periodic orbits with q reflections. A convex caustic is an invariant curve for the billiard map of the domain. Our result is thus: If Ω is a bounded smooth plane domain which is isospectral to a nearly circular ellipse of eccentricity ε, then Ω is ε-nearly circular in C^n for every $n \in \mathbb{N}$ (in particular it must be strictly convex) and Ω is rationally integrable.

The microlocal part of the proof is the study of the wave trace

$$w_\Omega(t) := \mathrm{Tr}\cos t\sqrt{\Delta_\Omega}.$$

It is well known that $w_\Omega(t)$ is a tempered distribution on $\mathbb{R}$ and that the positive singularities of w_Ω can only occur for $t \in \mathcal{L}(\Omega)$, the length spectrum (that is, the closure of the set of lengths of closed billiard trajectories). We study the singularities at closed trajectories of type $\Gamma(1,q)$, that is, with winding number 1 and with q bounces (reflections) off the boundary $\partial\Omega$. We denote the set of lengths of such closed trajectories by $\mathcal{L}_{1,q}(\Omega)$. For each q, the contribution to $w_\Omega(t)$ of closed trajectories $\Gamma(1,q)$ is denoted by $\hat{\sigma}_{1,q}$. A key ingredient in the proof is that a special type of oscillatory integral representation of $\hat{\sigma}_{1,q}$, introduced by Melrose–Marvizi for general convex domains for sufficiently large q, is in fact valid for all $q \geqslant 2$ for almost circular domains.

Using this representation, it is shown that the possible lengths of closed orbits have a "band-gap" structure, consisting of a union of bands $[t_q, T_q]$ of lengths for $\Gamma(1,q)$ orbits, separated by much larger gaps. The bands collapse to a single point if the domain is an ellipse, and therefore for any isospectral domain. Hence any domain isospectral to an ellipse of small eccentricity is rationally integrable. Alternatively, the phase $L_q(s)$ is constant for the ellipse and an analysis shows that any domain with asymptotically equivalent Melrose–Marvizi integrals must also have a constant phase.

General Relativity

Finally, we briefly describe the microlocal aspects of the Hintz–Vasy proof of global nonlinear stability of slowly rotating Kerr–de Sitter black holes, and the Häfner–Hintz–Vasy proof of linear stability of slowly rotating Kerr black holes. The Kerr family $g_{m,a}$ of black holes is a family of solutions of the vacuum Einstein equations $\mathrm{Ein}(g) := \mathrm{Ric}(g) - \frac{1}{2}R_g g = 0$, where g is a Lorentz metric on a 4-dimensional spacetime $M \simeq \mathbb{R}_{t_*} \times (0,\infty)_r \times \mathbb{S}^2$ and $b = (m, a)$ are mass and angular momentum parameters. "Slowly rotating" means that a is nearly 0. When $a = 0$, the Kerr metric is the Schwarzschild metric. Kerr–de Sitter spacetimes are solutions of $\mathrm{Ein}(g) = \Lambda g$, where Λ is the cosmological constant.

Nonlinear stability means that if (h,k) are smooth initial data on a spacelike (Cauchy) hypersurface Σ_0 (satisfying constraint equations), if (h,k) are close to the data (h_0,k_0) of a Schwarzschild–de Sitter spacetime, then there exists a solution g_b of $\mathrm{Ein}(g) = \Lambda g$ with the given Cauchy (initial) data (h,k) and with black hole parameter b close to b_0 such that $g - g_b = O(e^{-\alpha t_*})$: that is, g decays exponentially fast to the Kerr–de Sitter metric g_b. Linear stability refers to the linearization δL of L of the Einstein functional $\mathrm{Ein}(g)$ around the solution and means that solutions of the initial value problem for $L(\delta g) = 0$ decay (at a rate t_*^{-2} to the sum of a

Focus on the Scientist: Richard Melrose

Richard Melrose, a Chern Professor in the Microlocal Analysis program, is one of the key architects of this subject, and in his distinguished career has cut a wide swath across many parts of linear PDE, geometric analysis, differential geometry, and topology.

Richard was born in Sydney and grew up in Hobart, Tasmania. After attending the University of Tasmania, he took an honors year at the University of New South Wales in Canberra, and then moved to Cambridge University, where he completed his Ph.D. thesis in 1974 with Gerard Friedlander. After a year visiting MIT and another year back in Cambridge, he moved permanently to MIT. He assumed the Simons Professorship of Mathematics from 2006–2016.

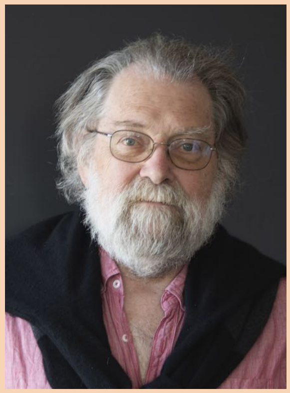


Photo: Lisa Jacobs

Richard Melrose

Richard's early career coincided with the great flourishing of microlocal analysis in the 1970s, and his first groundbreaking work concerned the propagation of singularities of solutions of the wave equation in the presence of boundaries. This led to his solution of the Lax–Phillips conjecture in scattering theory and to many other great advances in scattering and spectral theory and other parts of PDE. One of the things for which Richard is best known is his creation and fundamental role in the development of a subject now called geometric microlocal analysis. In fact, the paper that really started this field was one he wrote with Gerardo Mendoza during his semester-long stay at MSRI in the spring of its first year, 1983.

This set of ideas has led, in the hands of Richard and his many students and collaborators, to comprehensive results about elliptic and parabolic operators on stratified spaces, a very detailed and beautiful geometric understanding of propagation phenomena at infinity in stationary scattering on asymptotically Euclidean spaces and their generalizations, a definitive treatment of index theorems and families index theorems for manifolds with boundary, and a systematic treatment of diffraction phenomena for nonlinear waves, and many other topics besides.

Perhaps the cardinal feature of Richard's mathematical work is his thoroughly geometric vision in a wide range of problems, and his incisive and clear-sighted ability to understand the inner nature of the problems on which he works. His energy remains undiminished and his recent and ongoing work includes forays into analysis on loop spaces, Weil–Petersson geometry, and a new theory of compactifications of Lie groups, amongst other projects.

Richard has had 34 graduate students, many now on the faculty of major universities around the world. Richard was awarded the Bôcher Prize from the AMS in 1984; he was an invited speaker at the 1978 ICM in Helsinki and delivered a plenary address at the 1990 ICM in Kyoto. Richard's two books, *The Atiyah–Patodi–Singer Index Theorem* and *Geometric Scattering Theory*, have guided many, as have his copious unpublished lecture notes and expository manuscripts.

— Rafe Mazzeo

linearized Kerr metric plus the Lie derivative of $g_{m,a}$ along certain Killing vector fields. Here, $t_* = t - r - 2m \log r$ is a time parameter whose level sets are, roughly, outgoing light cones.

Stability problems for Minkowski, Schwarzschild, and Kerr (–de Sitter) spacetimes have a long history in physics and mathematics, associated with the work of Christodoulou, Klainerman, Rodnianski, Dafermos, and many others. The proofs of nonlinear stability involve an interation scheme. In the earlier, not very microlocal work, the iteration had roughly the "bootstrap" form: Set up the Einstein evolution problem as an ODE (choosing a gauge); solve the initial value problem for a short time; estimate the terminal value; iterate infinitely often.

The microlocal strategy of Vasy, Hintz–Vasy, Häfner–Hintz–Vasy is global in time. First, they compactify the Lorentz manifold M as a manifold with corners, in the spirit of Melrose's b-manifold theory. They solve the global nonlinear problems on the compactified space using a Nash–Moser iteration scheme; they solve a linearized equation globally at each step rather than locally in time by the bootstrap approach.

Microlocal tools for solving hyperbolic equations globally on a background like Kerr–de Sitter space were introduced by Vasy in a famous paper on Fredholm theory for hyperbolic equations. For wave equations on Kerr–de Sitter spacetimes, there are additional complications due to the trapped photon sphere, that is, lightlike geodesics which do not tend to future infinity. A microlocal analysis of the trapped set for such spacetimes was given earlier by Wunsch–Zworski and by Dyatlov.

The compactification of M is a more refined one than the classical Penrose conformal compactification: the boundary hypersurfaces of the compactification are now manifolds with corners. A generalization of the Melrose b-calculus produces an algebra of pseudodifferential operators in which one can analyze the linearized operator L_g. All nonelliptic estimates of the linearized equation have the form: "regularity along incoming directions implies regularity along outgoing directions," where incoming/outgoing directions refer to those points in the characteristic set of L_g that, in the future/past direction along the Hamilton flow of the principal symbol $\sigma_2(L_g)$, tend to the subset of phase space of interest, for example, the trapped photon sphere in the vicinity of the black hole. The incoming regularity assumption is traced back all the way to the regularity of the initial data; it implies a global estimate for solutions of initial value problems for L_g. Working in a compactified setting makes it possible to prove the uniformity of such estimates.

National Math Festival Connects Today's Wonder to Future Learning

The 2019 National Math Festival drew roughly 12,000 children and adults to the Walter E. Washington Convention Center in Washington, DC, on Saturday, May 4, for a joyful celebration of math, its beauty, its playful sides, and its manifold applications in the world.

Presentations included a world-class dance performance by the **BARKIN/SELISSEN PROJECT** of New York City, a journey through primes with **Holly Krieger**, a 300-year lightning-quick review (revue) of musical and mathematical history by **Lillian Pierce**, an exploration of the mathematics of braiding with Dance Your Ph.D. contest winner **Nancy Scherich**, a dive into the humanism of math with **Francis Su**, a modernization of the stable marriage problem with **Annie Raymond**, and many more. National Science Foundation funding for basic mathematics research was evidenced through the distribution of an NSF "passport" attendees could have stamped at each talk, showing the funding lineage of nearly all speakers!

The Alfred P. Sloan Foundation Film Room showcased interactive math talks interspersed with short film excerpts. Among these were **John Urschel**'s take on the physics of football, and the **National Science Foundation**'s unveiling of the winners of the We Are Mathematics short film contest. (Winners are posted on YouTube; search "We Are Mathematics contest.")

School Preview Day

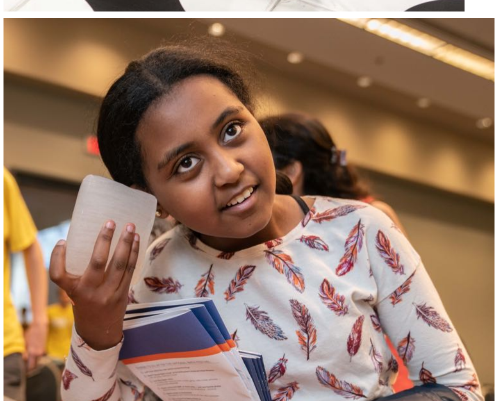
A School Preview Day on Friday, May 3, immersed 850 students in interactive, hands-on activities hosted by the **National Museum of Mathematics** (MoMath), the **Julia Robinson Mathematics Festival**, **MIND Research**, **Natural Math**, **Math-On-A-Stick**, the **Bridges Organization**, and the **Young People's Project**. These ranged from bilingual family board games in English and Spanish, to giant maze mats on the floor, to athletic competitions (factoring!) in the Flagway Game, to rapid Rubik's Cube solving, to art-making, puzzle-solving, and much more. In addition, roughly 600 students in class groups took advantage of the invitation on Saturday for the public festival.

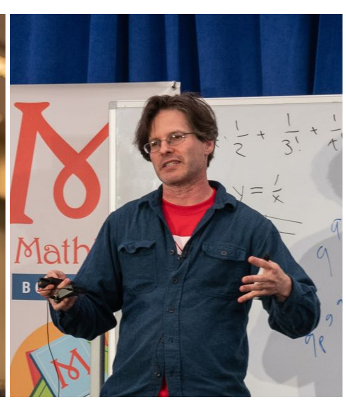
Both the public festival and the School Preview Day offered opportunities for Title I schools to register school groups and apply for donations to underwrite transportation costs, and many of the Friday and Saturday school attendees came from these schools. Title I schools enroll a high number or percentage of students from low-income families.

Curiosity that Sparks Long-Term Engagement

More than 85% of attendees were new to the NMF. Attendees came from an estimated 22–34 different US states. And those that came took away a long-term perspective on their math enjoyment.

More than 80% of attendees went on to discuss ideas from the NMF with friends or family who were not at the festival, and kept thinking about things they learned that day. About three-quarters picked up resources for future math enjoyment during the NMF. About two-thirds sought out further resources on a subject encountered at the festival.





Counterclockwise from page 8, top left — the Flagway™ Game; Marcus du Sautoy with David Eisenbud; MoMath's Tetra Truchet; climate math and ice cores; Mathical author Richard Schwartz; Bridges Math–Art Exhibit; 2019 NMF presenters; Lillian Pierce; the BARKIN/SELISSEN PROJECT; School Preview Day; Math Circle activities.

The Make or Take Spiral

As part of the NMF's aim to spark curiosity and convert that initial interest into long-term engagement with mathematics, a special focus of the 2019 NMF was the new Make or Take Spiral, which featured make-and-take activities for all ages as well as take-home resources for families and educators, opportunities to sign up to participate in math activities close to home, and guidance on plugging into high-quality, fun online clubs, activities, and other resources in recreational mathematics. Visitors made full use of this aspect of the festival.

We are grateful to the 23 math organizations who made this area of the festival come alive, among them the **Association for Women in Mathematics** (AWM), the **Benjamin Banneker Association** and **National Association of Mathematicians** (NAM), **NOVA**, the **Mathematical Association of America** (MAA), the **National Council of Teachers of Mathematics** (NCTM), **ThinkFun Games**, the **Erikson Institute Early Math Collaborative**, **Development and Research in Early Math Education** (DREME), **WGBH** with preschool games, apps, and activities, **Math Monday**, several local **Math Circles**, and students and faculty from **Ithaca College** and **Marymount University** with geometric balloon bending.

Across the Country

Math play was not limited to Washington, DC. In fact, more than 80 science centers in more than 40 US states hosted Geometric Bubble workshops reaching a total of more than 26,000 persons on Saturday, May 4. We are grateful to the Association of Science-Technology Centers (ASTC) and Zometool for making this day of nationwide math possible!

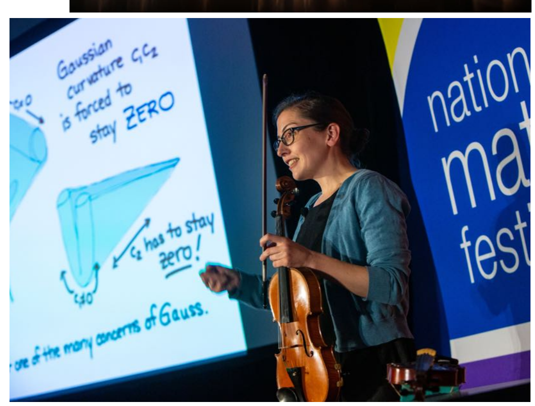
MSRI aims to keep the energy of the National Math Festival hopping year-round. One way we do this is to collect some of the very best fun family math resources at More Math! on the NMF web site. The offerings comprise a total of 100+ posts featuring 140 resources, with a few new posts added each month. Posts can be sorted by age band (ranging in ages from 2–18+) and activity type (Arts & Crafts, Books, Film & Video, Get Involved, Learn & Explore, Puzzles & Games).

Ways to Stay Involved

We hope you'll check out More Math! at nationalmathfestival.org/more-math. If you have resources to recommend, please email mathfestival@msri.org. To receive future National Math Festival news and updates, subscribe to the e-newsletter at www.tinyurl.com/nmfnews.

The 2019 National Math Festival was organized by the Mathematical Sciences Research Institute in cooperation with the Institute for Advanced Study and the National Museum of Mathematics (MoMath), with the generous support of the Simons Foundation; the Alfred P. Sloan Foundation; Eric and Wendy Schmidt; the National Science Foundation; the Gordon and Betty Moore Foundation; Irwin and Joan Jacobs; the Kavli Foundation; the American Mathematical Society; the Charles and Lisa Simonyi Fund for Arts and Sciences; Educational Testing Service; and Northrop Grumman.

2019 NMF Presenters — Front row, left to right: Annie Raymond, Nancy Scherich, Emily Riehl, James Tanton. Second row: Lillian Pierce, Amelia Taylor, Holly Krieger, Mark Mitton. Third row: Francis Su, John Urschel, Suzanne Weekes, Avi Wigderson. Not pictured: Marcus du Sautoy, Mary Lou Zeeman.



Holomorphic Differentials in Mathematics and Physics

(continued from page 1)

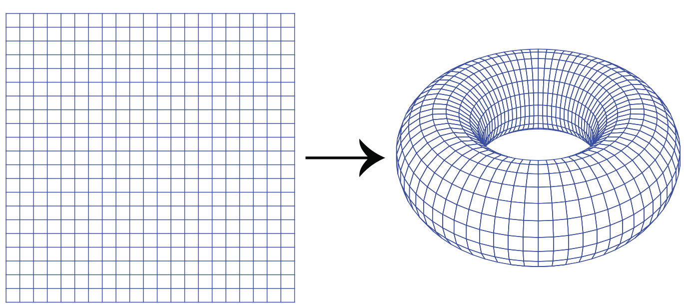
A Kaleidoscope of Holomorphic Differentials

The program on Holomorphic Differentials in Mathematics and Physics brings together people from many disparate areas in which holomorphic differentials play a role. Comparing and contrasting the different perspectives enriches our understanding and allows us to answer old questions and formulate new directions for the future.

In the following we describe a few of the appearances of holomorphic differentials, focusing on the areas that are key to the program.

Billiards and Abelian Differentials

A simple way to obtain a Riemann surface is to take a Euclidean polygon and glue opposite sides. If we start with a square, we get a torus, and if we start with an octagon, we get a surface with two holes, a genus 2 surface. Note that when we glue the square together, the grid on it continues to look the same everywhere, whereas on the genus 2 surface, twelve squares of the grid come together at one point, giving an angle of $12 \times \pi/2 = 6\pi$, though every other point looks normal.



Identifying opposite sides of a square to make a torus.

Since we are gluing sides by translations, maps of the form $z \mapsto z + c$ which are holomorphic maps, we get a holomorphic structure on these surfaces — but in fact, since $d(z+c) = dz$, these surfaces come equipped with a holomorphic 1-form, which, in local coordinates, is of the form $f(z)dz$. The zeros of this form correspond to points with excess total angle; an angle of $2\pi(n+1)$ corresponds to a zero of order n, so the genus 2 picture above has a zero of order 2. Such a holomorphic 1-form is also called an Abelian differential. They play an important role in understanding the dynamics of billiards in rational polygons.

More generally, we can consider holomorphic k-differentials, locally of the form $f(z)(dz)^k$, with $f(z)$ holomorphic. These arise naturally on surfaces obtained from polygons by gluing sides with maps of the form $z \mapsto \zeta_k z + c$, where ζ_k is a k-th root of unity, equivalently, with rotations of order k and translations. The case $k = 2$ is of particular interest; these are known as holomorphic quadratic differentials.

An important tool in studying Abelian and quadratic differentials is the action of $SL(2, \mathbb{R})$ on the space of Abelian differentials. This action is easy to describe from the point of view of flat surfaces obtained by gluing planar polygons: just let $SL(2, \mathbb{R})$ act on the plane! Nevertheless, its dynamics are surprisingly subtle; the characterization of the orbit closures of this $SL(2, \mathbb{R})$-action is one of the results for which Maryam Mirzakhani was awarded the Fields Medal, and Alex Eskin the Breakthrough Prize in Mathematics.

Higgs Bundles, Quadratic Differentials, and Higher Teichmüller Spaces

A holomorphic k-differential is a holomorphic section of the k-th power of the canonical bundle K of the Riemann surface. In this incarnation, holomorphic differentials arise in the theory of Higgs bundles. A Higgs bundle is a pair (E, Φ) where E is a rank n holomorphic bundle and Φ, the Higgs field, is a holomorphic bundle map $E \to E \otimes K$. The coefficients of the characteristic polynomial of Φ are holomorphic differentials of degree $k = 1, \cdots, n$. The map which takes (E, Φ) to this tuple of holomorphic k-differentials is the Hitchin fibration.

In the case $n = 2$, Hitchin used a section of the Hitchin fibration to provide a new parametrization of Teichmüller space by the space of holomorphic quadratic differentials on a fixed Riemann surface X. The construction starts with the line bundle $K^{1/2}$ on a Riemann surface X, by which we mean a line bundle whose square is isomorphic to K. The dual of this line bundle, $K^{-1/2}$, is a square root of K^{-1}. Taking the rank two bundle $K^{-1/2} \oplus K^{1/2}$ to be E, the Higgs field Φ is then a map

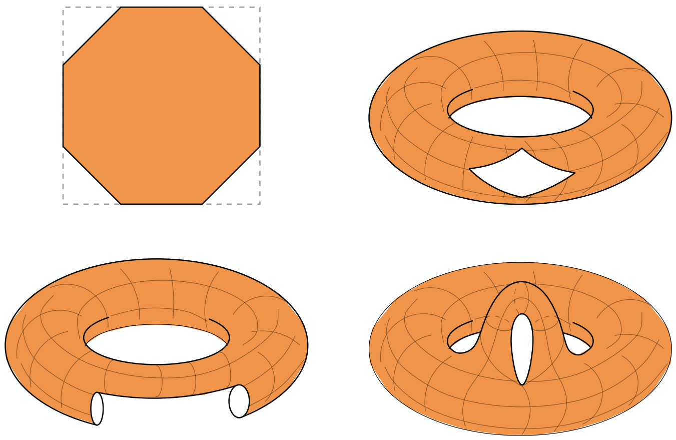
$$\Phi : K^{-1/2} \oplus K^{1/2} \to K^{1/2} \oplus K^{3/2}.$$

One can choose Φ to have the off-diagonal form $\Phi = \left(\begin{smallmatrix} 0 & c \\ q & 0 \end{smallmatrix}\right)$, where $c : K^{1/2} \to K^{1/2}$ and $q : K^{-1/2} \to K^{3/2}$. Holomorphic maps between line bundles over a compact manifold are constants, so c can be normalized to be 1. Any map of line bundles from L_1 to L_2 yields a holomorphic section of the line bundle $L_1^{-1} \otimes L_2$, so q is a section of K^2, that is, a quadratic differential.

This construction requires a fixed complex structure on S, the real surface underlying X. We can take this as a basepoint in Teichmüller space, the space parameterizing (marked) complex structures on S. Hitchin showed that a new hyperbolic metric on S can be constructed from the quadratic differential q and a bundle metric on $K^{1/2}$, provided that the induced bundle metric on $K^{-1/2} \oplus K^{1/2}$ solves a certain partial differential equation, now known as Hitchin's equation. In this way he obtained a new parameterization of Teichmüller space by quadratic differentials.

Hitchin's equation makes sense not only for the Higgs bundle we described above, but for any Higgs bundle (E, Φ), and it admits a unique solution provided that (E, Φ) obeys a certain algebro-geometric stability condition. More generally, for any Lie group G, one can define a notion of G-Higgs bundle. Solving Hitchin's equation for such a bundle produces a flat connection with holonomies in G. Thus one obtains a diffeomorphism between the space of G-Higgs bundles and the space of flat G-connections, in turn identified with the variety parameterizing reductive representations $\pi_1(X) \to G$. This is the celebrated non-Abelian Hodge correspondence.

Now we can return to holomorphic differentials: For general G, one considers not just quadratic differentials but tuples of holomorphic



Identifying opposite sides of an octagon to make a genus 2 surface.

differentials of various degrees; for each such tuple one gets a Higgs bundle; as we let the holomorphic differentials vary, these Higgs bundles sweep out a section of the Hitchin fibration, the Hitchin section. Via the non-Abelian Hodge correspondence, the Hitchin section gives rise to new phenomena in the representation variety of the fundamental group of the surface into G. The image of the Hitchin section forms the Hitchin component, a connected component, consisting entirely of discrete and faithful representations. In many respects the Hitchin component resembles Teichmüller space.

The study of the Hitchin component, and of other higher (rank) Teichmüller spaces, such as the space of maximal or positive representations, is a very active area. So far it is mostly Thurston's hyperbolic point of view on Teichmüller space which has been generalized to these higher Teichmüller spaces; the complex analytic viewpoint is still mainly obscure in these cases. Nevertheless, when the Lie group G has rank 2, there are mapping-class-group invariant parameterizations of the Hitchin component (and in a similar way of the space of maximal representations) by pairs of a conformal structure and a holomorphic k-differential. Recently flat structures on surfaces, which are determined by Abelian differentials, also seem to make their appearance in new compactifications of these higher Teichmüller spaces at infinity. There are still many open questions and relations to be discovered.

Quantum Field Theory

Recently a new role for Riemann surfaces and holomorphic differentials has emerged in high energy physics. The starting point for this story is a certain six-dimensional quantum field theory $\mathfrak{X}(\mathfrak{g})$, depending on a Lie algebra $\mathfrak{g}$ (for example, $\mathfrak{g} = \mathfrak{sl}(N)$, the algebra of $N \times N$ traceless matrices). The field theories $\mathfrak{X}(\mathfrak{g})$ were discovered in the mid-1990s and remain rather mysterious to the present day. In 2009 physicists proposed a thought experiment: suppose that the universe is described by one of the field theories $\mathfrak{X}(\mathfrak{g})$, and choose the spacetime to be of the form $M_6 = X \times M_4$. If X is much smaller than M_4, observers living in this hypothetical spacetime will not see X directly; they will perceive their universe to be M_4. Nevertheless, the laws of physics they will observe in M_4, governed by a four-dimensional field theory $S(X, \mathfrak{g})$, are intimately tied up with the structure of X.

For example, one can consider the couplings in $S(X, \mathfrak{g})$ — fundamental parameters determining the strength of the various interactions in the theory, analogous to the fine-structure constant in our universe. It turns out that these couplings are most naturally considered not as numbers, but rather as coordinates on the Teichmüller space of X. Similarly, many other physical phenomena in $S(X, \mathfrak{g})$ have translations into the geometry of X, and vice versa; a few examples follow:

Teichmüller space of X	$\leftrightarrow$	Coupling space of $S(X, \mathfrak{g})$
Holomorphic differentials on X	$\leftrightarrow$	Vacuum states in $S(X, \mathfrak{g})$ on $M_4 = \mathbb{R}^4$
$\mathfrak{g}$-Higgs bundles over X	$\leftrightarrow$	Vacuum states in $S(X, \mathfrak{g})$ on $M_4 = \mathbb{R}^3 \times S^1$
Billiard trajectories between singularities on X	$\leftrightarrow$	Supersymmetric particles in $S(X, \mathfrak{sl}(2))$
Webs of trajectories on X	$\leftrightarrow$	Supersymmetric particles in $S(X, \mathfrak{sl}(N))$
Loops and spin-networks on X	$\leftrightarrow$	Line operators in $S(X, \mathfrak{sl}(N))$
Mapping class group of X	$\leftrightarrow$	Duality group of $S(X, \mathfrak{g})$

This thought experiment is not yet rigorous mathematics; nevertheless, it has turned out to be a fertile source of mathematical ideas. A few examples are unexpected connections between instanton integrals on $\mathbb{R}^4$ and conformal field theory on surfaces, a new scheme for understanding the hyperKähler metrics on moduli spaces of Higgs bundles, and new insights in the theory of Donaldson–Thomas invariants of Calabi–Yau threefolds. In the other direction, techniques in dynamics of flat surfaces have led to the solution of particle counting problems in the physical theories $S(X, \mathfrak{sl}(2))$, and more generally, ideas from cluster algebra, higher Teichmüller theory, dynamics and other areas have been essential inputs in the discovery of new phenomena in the field theories $S(X, \mathfrak{g})$.

Named Positions, Fall 2019

MSRI is grateful for the generous support that comes from endowments and annual gifts that support faculty and postdoc members of its programs each semester.

Chern, Eisenbud, and Simons Professors

Pierre Albin, University of Illinois at Urbana-Champaign
Nalini Anantharaman, Université de Strasbourg
Vladimir Fock, Université de Strasbourg
Colin Guillarmou, Université de Paris XI (Paris-Sud)
Rafe Mazzeo, Stanford University
Richard Melrose, Massachusetts Institute of Technology
Andrew Neitzke, University of Texas, Austin
Andras Vasy, Stanford University
Richard Wentworth, University of Maryland
Steve Zelditch, Northwestern University
Anton Zorich, Université de Paris VII (Denis Diderot)

Named Postdoctoral Fellows

McDuff: Dylan Allegretti, University of Sheffield
V. Della Pietra: Laura Fredrickson, Stanford University
Gamelin: Katrina Morgan, Northwestern University
Viterbi: Hui Zhu, Université de Paris XI
Uhlenbeck: Xuwen Zhu, University of California, Berkeley

Focus on the Scientist: Anna Wienhard

Anna Wienhard was born in 1977 in Giessen, Germany; at age seven her family moved to Köln as her father, an experimental physicist, obtained a position at the Max Planck Institute. Throughout her high school studies, Anna's interests were multifaceted: she enjoyed history, Latin, mathematics, and physics, as well as social projects in Egypt and Brazil.

HITS, Gülay Keskin

Anna Wienhard

Entering university, Anna was torn between studying mathematics and theology. Confronted with this choice she decided — and this is a character trait of hers — to pursue both goals with the same energy! In 2000 she obtained her undergraduate "Diplom" with a thesis on the spectrum of the Laplacian where she came in contact with Mike Wolf's thesis and hence with Teichmüller theory. At about the same time she obtained her Diplom in theology.

She then began a Ph.D. with Werner Ballmann and developed a strong interest in bounded cohomology. Keen to travel, she came to spend a few months at ETH in Zurich in September 2002. The result is that in March 2003, she coauthored a *Comptes Rendus* note that essentially laid the foundations for the study of the geometric aspects of maximal representations. Her thesis, defended in 2004, introduced many new concepts like tight embeddings and weakly maximal representations.

In 2005 as a postdoc in Basel with Alessandra Iozzi, she coorganized the landmark Strasbourg–Basel conference on Bounded Cohomology, Harmonic Maps, and Higgs Bundles during which her collaboration with Olivier Guichard began. In a series of very influential papers, Anna and Olivier developed a geometric picture of Anosov representations as Kleinian groups in higher rank.

Together with like-minded colleagues she founded the GEAR network that grew to encompass over 80 nodes in all continents; it had a profound impact on the subject of higher Teichmüller theory and helped create a sense of community.

Since 2012 she has been a professor in Heidelberg, where she lives with her husband Daniel, a physicist, and their four children. Since 2015 she has also been group leader in the Heidelberger Institut für Theoretische Studien. Among her current interests is a notion of positivity that is designed to unify all phenomena pertaining to higher Teichmüller theory.

— Marc Burger

Call for Proposals

All proposals can be submitted to the Director or Deputy Director or any member of the Scientific Advisory Committee with a copy to proposals@msri.org. For detailed information, please see the website msri.org/proposals.

Thematic Programs

The Scientific Advisory Committee (SAC) of the Institute meets in January, May, and November each year to consider letters of intent, pre-proposals, and proposals for programs. The deadlines to submit proposals of any kind for review by the SAC are **March 1**, **October 1**, and **December 1**. Successful proposals are usually developed from the pre-proposal in a collaborative process between the proposers, the Directorate, and the SAC, and may be considered at more than one meeting of the SAC before selection. For complete details, see tinyurl.com/msri-progprop.

Hot Topics Workshops

Each year MSRI runs a week-long workshop on some area of intense mathematical activity chosen the previous fall. Proposals should be received by **March 1**, **October 1**, and **December 1** for review at the upcoming SAC meeting. See tinyurl.com/msri-htw.

Summer Graduate Schools

Every summer MSRI organizes several two-week long summer graduate workshops, most of which are held at MSRI. Proposals must be submitted by **March 1**, **October 1**, and **December 1** for review at the upcoming SAC meeting. See tinyurl.com/msri-sgs.

Call for Membership

MSRI invites membership applications for the 2020–21 academic year in these positions:

Research Members	by December 1, 2019
Postdoctoral Fellows	by December 1, 2019

In the academic year 2020–21, the research programs are:

Random and Arithmetic Structures in Topology
Aug 17 – Dec 18, 2020

Organized by Nicolas Bergeron, Jeffrey Brock, Alexander Furman, Tsachik Gelander, Ursula Hamenstädt, Fanny Kassel, Alan Reid

Decidability, Definability, and Computability in Number Theory
Aug 17 – Dec 18, 2020

Organized by Valentina Harizanov, Maryanthe Malliaris, Barry Mazur, Russell Miller, Jonathan Pila, Thomas Scanlon, Alexandra Shlapentokh, Carlos Videla

Mathematical Problems in Fluid Dynamics
Jan 19 – May 28, 2021

Organized by Thomas Alazard, Hajer Bahouri, Mihaela Ifrim, Igor Kukavica, David Lannes, Daniel Tataru

MSRI uses **MathJobs** to process applications for its positions. Interested candidates must apply online at mathjobs.org. For more information about any of the programs, please see msri.org/programs.

Named Postdocs — Fall 2019

Gamelin

Katrina Morgan is the Gamelin postdoc in this fall's Microlocal Analysis program. She received her Ph.D. this past spring at the University of North Carolina at Chapel Hill under the direction of Jason Metcalfe. Her research has concentrated on the relationship between the long-time behavior of solutions to the wave equation and the underlying geometry in an asymptotically flat setting. Asymptotically flat spacetimes arise in general relativity, which has motivated many mathematical questions about such wave behavior. Katrina was an undergraduate at Rice University, majoring in cognitive science — an interdisciplinary program in computer science, biology, psychology and linguistics. She was interested in math in college and high school and decided to switch fields and pursue a Ph.D. in mathematics. Another important part of her career has been to interest high school girls in math. In 2016 she co-founded Girls Talk Math, a summer day camp for high school girls interested in math at UNC (girlstalkmath.com). The program expanded to the University of Maryland in 2018. After this semester's program, Katrina will go to Northwestern as an RTG Postdoc under the mentorship of Jared Wunsch. *The Gamelin postdoctoral fellowship was created in 2014 by Dr. Ted Gamelin, Emeritus Professor of the UCLA Department of Mathematics. The Gamelin Fellowship emphasizes the important role that research mathematicians play in the discourse of K-12 education.*

Katrina Morgan

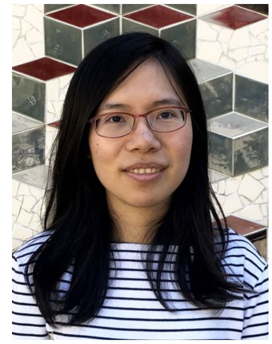
Vincent Della Pietra

Laura Fredrickson is the Vincent Della Pietra Fellow in this fall's program on Holomorphic Differentials in Mathematics and Physics. She obtained her bachelor's degree in mathematics from the University of California, Irvine, and her Ph.D. from The University of Texas at Austin under the supervision of Andy Neitzke. She is currently a Szegö Assistant Professor at Stanford. Laura's work deals with solutions of Hitchin's self-duality equations. This theory could be thought as a far-reaching generalization of the fact that harmonic functions are the real part of holomorphic functions. One of the main issues is to understand the asymptotics of the solution (in our trivial example, the harmonic function) when the Higgs field (here the holomorphic function) goes to infinity. Laura has obtained crucial results in this extremely difficult area. *The Vincent Della Pietra fellowship was established in 2017 by the Della Pietra Foundation. Vincent received his Ph.D. in mathematical physics from Harvard University. He is a partner at Renaissance Technologies, co-founder of the Della Pietra Lecture Series at Stony Brook University, and a board member of PIVOT.*

Laura Fredrickson

Uhlenbeck

Xuwen Zhu is the Karen Uhlenbeck Postdoctoral Fellow in this semester's Microlocal Analysis program. In 2015 Xuwen received her Ph.D. in mathematics from the Massachusetts Institute of Technology, under the supervision of Richard Melrose. Her research interests are in partial differential equations motivated by problems from geometry and mathematical physics, in the context of singular spaces. She obtained, with Melrose, a precise description of the cusp degenerations of hyperbolic metrics on Riemann surfaces, which led to improved understanding of the boundary structure of the Deligne–Mumford–Knudsen compactification of the Riemann moduli spaces. Xuwen has also obtained new results, with Mazzeo, about moduli spaces of constant curvature metrics with conical singularities. *The Uhlenbeck fellowship was established by an anonymous donor in honor of Karen Uhlenbeck, a distinguished mathematician and former MSRI trustee. She is a member of the National Academy of Sciences and a recipient of the 2019 Abel Prize, the AMS Leroy P. Steele Prize, and a MacArthur "Genius" Fellowship.*



Xuwen Zhu

McDuff

Dylan Allegretti is the McDuff Postdoctoral Fellow for the program on Holomorphic Differentials in Mathematics and Physics. His work focuses on cluster varieties and their relationships with important ideas in mathematical physics such as quantum Teichmüller theory, wall crossing phenomena, and spaces of stability conditions on triangulated categories. The theory of cluster varieties, introduced by Fock and Goncharov, is a geometric framework that uses combinatorial ideas from cluster algebras. It provides a powerful tool to study moduli spaces of local systems. Dylan earned undergraduate degrees in both mathematics and physics from the University of Chicago. In 2016, he was awarded a Ph.D. in mathematics from Yale University under the supervision of Alexander Goncharov. From 2016–19 he was a Research Associate at the University of Sheffield where he worked with Tom Bridgeland, and beginning next year he will be a postdoctoral fellow at the University of British Columbia. *The McDuff fellowship was established by an anonymous donor in honor of Dusa McDuff. She is an internationally renowned mathematician, a member of the National Academy of Sciences, and a recipient of the AMS Leroy P. Steele Prize (2017). She is also currently a trustee of MSRI.*

Dylan Allegretti

Puzzles Column

Joe P. Buhler and Tanya Khovanova

Many mathematicians love brain teasers. At David Eisenbud's strong suggestion, Elwyn Berlekamp and I (JB) started writing the Emissary Puzzles Column when I arrived as Deputy Director in the fall of 1999. The idea was to capture contagious bits of mathematics brought to MSRI by its steady stream of visitors.

This has been a vivid and delightful experience for me, and I have many fond memories of discussions, in person and in email, about which problems to include, how to solve them, and (especially) how to phrase them. Elwyn was never shy about criticizing anything that he viewed as imprecise, inelegant, or unartfully phrased, but he was also full of delight at new and interesting ideas. This interchange continued unabated up to the very last few weeks of his life.

This column is dedicated to the memory of Elwyn, and his love of puzzles. We include a diverse selection of puzzles that evoke Elwyn, in one way or another. One of them is an open question, due to Elwyn, that arose out of a "hat puzzle" in the column; Elwyn would really like someone to solve it!

Mathematicians in hats, a frequent subject of the Puzzles Column. David Eisenbud with Elwyn Berlekamp in October 2015.

Puzzles for Elwyn

1. Prove that the order of any automorphism of a finite group of order $n > 1$ has order less than n.

Comment: DO NOT WORK ON THIS PROBLEM! This was the last problem of the Fall 1999 column. An eminent member of the UC Berkeley mathematics department proposed it and said that it was a bit on the harder side, despite the simplicity of the question. In fact, it is extremely difficult; it can be done by (slightly tedious) enumeration of cases starting with the classification of simple groups, but two solutions in print are both unbelievably intricate and dense (to any non-group-theorist); asking the Emissary reader to solve this was "cruel and unusual." For the next twenty years, we solved all problems before including them. (The alert reader will note that our final problem here contravenes this stricture, albeit overtly.)

2. The decimal representation of 2^{29} has nine distinct digits. Which digit is missing?

Comment: This problem was proposed by the aforementioned UCB mathematician, and we were happy that it was on the other end of the difficulty spectrum. When it was described to Elwyn orally, he instantly said that he didn't like the problem because you could just plug it into a calculator and be done with it. About 10 seconds later, he had an "aha" moment and said that, no, he thought that it was a great problem!

3. Each of the twelve faces of a dodecahedron has a light that is also an on/off button. Pushing the light causes all five of the lights on the adjacent faces to switch state (go from on to off, or the reverse). Prove that any of the 2^{12} positions can be obtained from any other by a suitable sequence of button pushes.

Comment: Elwyn introduced the famous rectangular grid "lights out" puzzle at Bell Labs in the 1960s. The idea has been generalized in many directions; the dodecahedral version above was commercially available more than 20 years ago, and other variants can be found either in game stores, or in math papers. (For example: Martin Kreh (2017), "'Lights Out' and Variants," The American Mathematical Monthly, 124:10, 937-950.)

4. Find three random variables each uniformly randomly distributed on [0,1] such that their sum is constant. That is, find a probability distribution on the intersection of $x+y+z=1$ with the unit cube in 3-space such that the three "coordinate projections" onto [0,1] are uniform.

Comment: Numerous solutions are possible, and this question generated more correspondence and greater diversity of solutions than any other Emissary puzzle has.

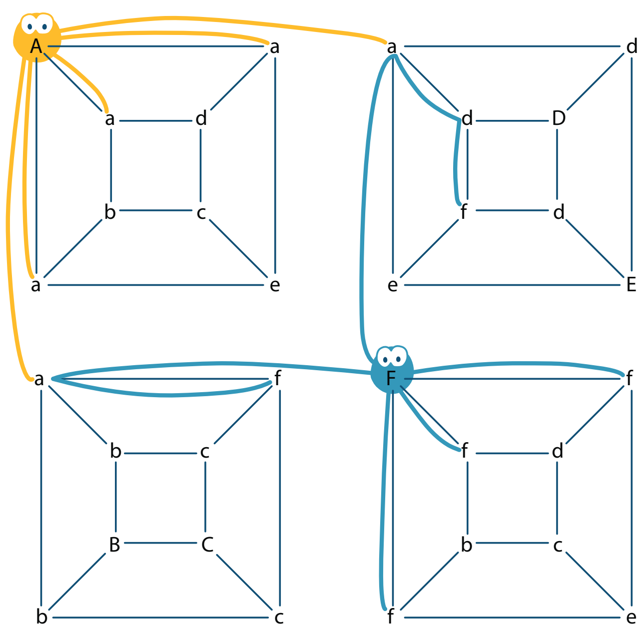
5. (i) Alice and Bob each have \$100 and a biased coin that comes up heads 51% of the time. On a signal, they each start flipping their coins once a minute, betting \$1 on the result of the flip. Alice bets on heads; poor Bob bets on tails. As it happens, they both eventually go broke. In that case, who is more likely to have gone broke first?

(ii) As above, but this time Alice and Bob bet on the result of the same coin flip (say, flipped by a referee). Again, assume both eventually go broke. Who is more likely to have gone broke first?

Comment: Elwyn was always a great fan of Peter Winkler's work, perhaps especially including his "recreational" mathematical problem books. This problem was contributed by Peter for the sake of this column.

6. Let H_n be the n-dimensional hypercube graph, whose vertices are (labeled by) the 2^n binary strings of length n, with edges joining vertices whose labels differ in exactly one coordinate. A *spider* in this graph consists of a central vertex c together with n paths, called legs, that start at c. A set of spiders covers H_n if no legs share an edge (though they are allowed to cross at vertices) and every vertex is either a central vertex of some spider, or an endpoint of a spider leg. Prove that there is a collection of m spiders that cover with exactly

$$m = \lceil \frac{2^n}{n+1} \rceil.$$



Note that this is the minimum possible number, since each spider accounts for $n+1$ points (its center and each endpoint of a leg) and there are 2^n points.

The figure shows a hypercube in five dimensions, where some edges are omitted so as not to clutter the diagram. Namely, every vertex has two extra edges connecting it to its sisters in the horizontally and vertically adjacent 3-cubes. A covering by six spiders is illustrated — the spiders are at the upper case letters, and the endpoints of their legs are at corresponding lower case letters. It is almost a covering by Hamming balls (one example being spider A), except that spider F has one leg of length two and one of length three.

Comment: The "hat problem" in the Fall 2001 column was referenced in a *New York Times* article ("Why Mathematicians Now Care About Their Hat Color," April 10, 2001) and has been cited in several academic papers; it is due to Todd Ebert. An earlier paper by Aspnes, Beigel, Furst, and Rudich had a "voting" problem that has a very similar solution (using perfect Hamming codes). The latter can be expressed as a "majority hats problem," and the truth of the above conjecture, which Elwyn made after he had settled many cases, would imply that the solution to the optimal probability for the voting problem is unexpectedly explicit whereas the size of the solution for the original hat game seems to be unknowable in polynomial time (except, roughly, for $n = 2^k$). All this will be explained in detail in the problem solutions.

Forthcoming Workshops

Nov 18–22, 2019: *Holomorphic Differentials in Mathematics and Physics*

Dec 9, 2019: *Symposium in Honor of Julia Robinson's 100th Birthday*

Jan 23–24, 2020: *Connections for Women: Quantum Symmetries*

Jan 27–Jan 31, 2020: *Introductory Workshop: Quantum Symmetries*

Feb 6–7, 2020: *Connections for Women: Higher Categories and Categorification*

Feb 10–Feb 14, 2020: *Introductory Workshop: Higher Categories and Categorification*

Mar 11–13, 2020: *CIME 2020: Today's Mathematics, Social Justice, and Implications for Schools*

Mar 16–20, 2020: *Tensor Categories and Topological Quantum Field Theories*

Mar 23–27, 2020: *(∞, n)-Categories, Factorization Homology, and Algebraic K-theory*

May 4–8, 2020: *Hot Topics: Optimal Transport and Applications to Machine Learning and Statistics*

Jun 13–Jul 26, 2020: *MSRI-UP 2020: Branched Covers of Curves*

Summer Graduate Schools

Jun 8–19, 2020: *Combinatorial and DG-Algebra Techniques for Free Resolutions (Tianjin, China)*

Jun 15–26, 2020: *Geometric Flows (Athens, Greece)*

Jun 15–26, 2020: *Algebraic Theory of Differential and Difference Equations, Model Theory and their Applications*

Jun 29–Jul 10, 2020: *New Directions in Representation Theory (AMSI, Brisbane, Australia)*

Jun 29–Jul 10, 2020: *Random Graphs*

Jun 29–Jul 10, 2020: *Séminaire de Mathématiques Supérieures 2020: Discrete Probability, Physics and Algorithms (Montréal, Canada)*

Jun 29–Jul 10, 2020: *Algebraic Curves (Hainan, China)*

Jun 29–Jul 10, 2020: *Foundations and Frontiers of Probabilistic Proofs (Zurich, Switzerland)*

Jul 6–17, 2020: *Metric Geometry and Geometric Analysis (Oxford, United Kingdom)*

Jul 13–Jul 24, 2020: *Sums of Squares Method in Geometry, Combinatorics and Optimization*

Jul 27–Aug 7, 2020: *Introduction to Water Waves*

For more information about any of these workshops, as well as a full list of all upcoming workshops and programs, please see msri.org/workshops.

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Q&A | Roger Strauch

***Roger Strauch** is co-founder and chairman of The Roda Group, an early stage venture capital firm based in Berkeley. An MSRI donor since the 1990s, he ended a term as President of the Board of Trustees earlier this year. Roger's family connections to the* Curious George *books inspired MSRI's Mathical Book Prize.*

Why is MSRI important to you?

It's been an association with a group of people who are natural, creative, serious, fun problem solvers and whose devotion to mathematics is both genuine and inspiring. I have a natural affinity to people who are mathematicians.

Why is that?

As an electrical engineer, my original field was signal processing, where you use mathematics to characterize signals for communications. I had a sense of the power and beauty of mathematics very early in my career. I'm also the son of an academic, a physics professor.

Was that significant?

It gave me a unique perspective. When the Cold War entered it was much more difficult for my father and his colleagues to attract research funding in particle physics. They were ill-equipped to justify their work to the American people. As a result, their funding more or less collapsed. I felt this would happen to mathematicians unless they seriously took their responsibility to learn how to communicate with the American people who provided 100% of the funding for the institute. I also thought it would be exciting to attract private resources to MSRI, which is the gold standard for how to go about conducting collaborative research in the scientific world.

Is it important to have both public and private funding?

MSRI will remain vibrant and have longevity precisely because there's a public-private partnership. The public feels that the private sector appreciates the value of MSRI and the private sector appreciates the public investment. It feels healthy, wealthy, and wise to have both.

But why support math research specifically?

Because math is the language used to create and communicate knowledge. Most, if not all, scientific fields involve mathematics. So we who appreciate that should help support it.

You're a longtime donor, why do you continue giving to MSRI?

When you donate to MSRI the fact is that you do make a difference practically. The management of this institute, David Eisenbud and Hélène Barcelo, have made outstandingly effective use of the capital that's been available to them, to support the next generation of talent, to support excellent research, and to support programs that present the power and beauty of mathematics to the public.