Rigidity Phenomena via Ergodic Theory

Based on j/w Uri Bader

Higher Rank vs. $\boxed{\text{Hyperbolic like}}$

$G = SL_n(k)$ $n \geq 3$ $k = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p[[t]] \ldots$

Theorem 1 let G be a simple Lie group $rk(G) \geq 2$
let $\Gamma < G$ be a lattice
let $H \curvearrowright \underset{\text{cpt.}}{M}$ be a convergence group.
then any $\rho : \Gamma \to H$ is elementary
i.e. $\rho(\Gamma)$ fixes $\mu \in \mathrm{Prob}(M)$.

Recall

Defn: $H \curvearrowright M$ cpt. is convergence action if
$H \curvearrowright M^{(3)} := \{(x_1, x_2, x_3) \in M^3 \mid x_i \neq x_j\}$ is proper

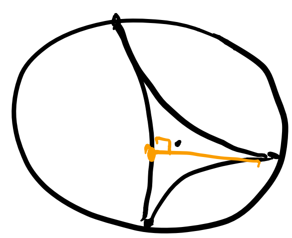
$L < H$ is elementary if L precompact, or
L fixes $x \in M$, or
L fixes $\{x, y\} \in M^{(2)}$.

Ex. - $H = SL_2(\mathbb{R})$ $M = \mathbb{P}^1_{\mathbb{R}}$

- $H < \mathrm{Isom}(\mathbb{H}^n_K)$ $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

- Gromov hyperbolic groups

- ...



Theorem 2 Let $G = G_1 \times \dots \times G_n$ $n \geq 2$.

$\Gamma < G$ lattice, $pr_i(\Gamma)$ dense in G_i $i=1,\dots,n$

$H \curvearrowright M$ convergence. $\rho : \Gamma \to H$ a hom.

either $\rho(\Gamma)$ is elementary

OR $\exists i$ + minimality

$\pi : G_i \to H$ s.t.

$$\begin{tikzcd} \Gamma \arrow[r, "\rho"] \arrow[d] & H \\ G \arrow[r] & G_i \arrow[u] \end{tikzcd}$$

Ex. - $SL_2(\mathbb{Z}[\sqrt{2}]) < SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

- $SL_2(\mathbb{Z}[\frac{1}{p}]) < SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p)$

- Burger-Mozes groups

Ergodic Theory – Boundaries

Γ any (countable group)

Measure class preserving / non-singular Γ-actions

Def $\Gamma \curvearrowright (X,\mu)$ $\quad \Gamma \times X \to X$ is measurable

$g_*\mu \sim \mu \quad \forall g \in \Gamma$, i.e. $\Gamma \to \mathrm{Aut}(L^\infty X)$

Defn a non-singular $\Gamma \curvearrowright (X,\mu)$ is

- ergodic if $1_E \in L^\infty(X,\mu)$ is Γ-invariant
  $\Rightarrow 1_E = \text{const}$, i.e. $1_E = 1_\emptyset$ or $1_E = 1_X$

- metrically ergodic if for any
  $\Gamma \curvearrowright (V,d)$ by isometries one has
  $f : X \to V$ (meas.) $\quad f(g\cdot x) = g\cdot f(x)$ (μ-a.e.) $\Rightarrow f = \text{const.}$ (μ-a.e.)

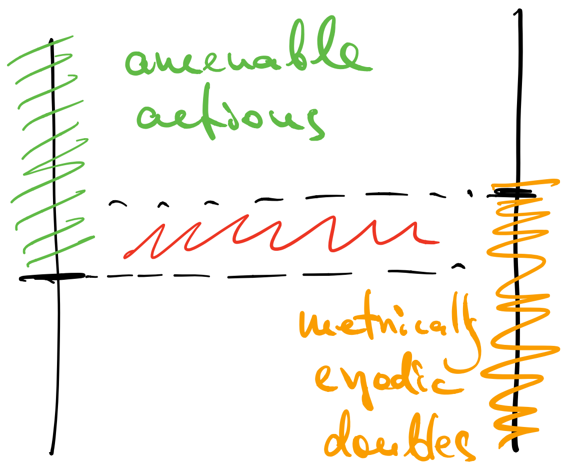
  $$\mathrm{Map}_\Gamma(X,V) = \{\text{const}_v \mid v \in V^\Gamma\}$$

- $\Gamma \curvearrowright (X,\mu)$ is amenable if for any
  $\Gamma \to \mathrm{Homeo}(M)$ $\quad M$ cpt.

  $$\mathrm{Map}_\Gamma(X, \mathrm{Prob}(M)) \neq \emptyset.$$

Defn A non-singular action $\Gamma \curvearrowright (B,\nu)$ is a <u>boundary</u> action if

- $\Gamma \curvearrowright (B,\nu)$ is amenable
- $\Gamma \curvearrowright (B\times B, \nu\times\nu)$ is metrically ergodic



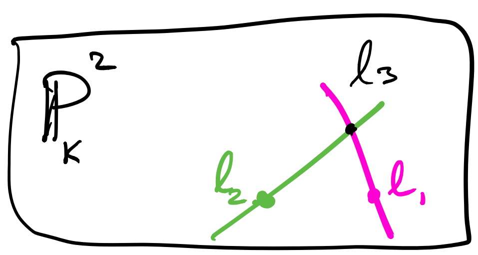
Thm Any Γ has Boundary Actions.

Prop: $\Gamma < G$ is a lattice, G - semi-simple $\Rightarrow$ $B = G/P$ Boundary.

Case $G = SL_3(k)$ or $PGL_3(k)$ $\quad k$ - local field

$$P = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \qquad A = \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & * \end{pmatrix}$$

$$G/P = \{ (\ell, \Pi) \mid \underset{\text{1-dim}}{\ell} \subset \underset{\text{2-dim}}{\Pi} \}$$



$$G/P \times G/P \underset{\text{a.e.}}{\overset{\supset}{=}} G/A = \{ (\ell_1, \ell_2, \ell_3) \mid \ell_1 \oplus \ell_2 \oplus \ell_3 = k^3 \}$$

Claim: $\Gamma < G$ lattice $\Rightarrow$ $B = G/P$ Γ-boundary

(i) $\Gamma \curvearrowright G/P$ amenable

(ii) $\Gamma \curvearrowright G/P \times G/P = G/A$ is metrically ergodic

Sketch of proof

(i) using amenability of P.

Given compact M and $\Gamma \to \mathrm{Homeo}(M)$ look at

$$\mathrm{Map}_\Gamma(G, \mathrm{Prob}(M)) \cong L^\infty(\Gamma\backslash G, \mathrm{Prob}(M))$$

$$L^1(\Gamma\backslash G, C(M))^* = L^\infty(\Gamma\backslash G, C(M)^*)$$

P acts on this convex compact by affine transf.

$\Rightarrow \exists$ P-fixed point.

$\rightsquigarrow$ P-invariant $\mathrm{Map}_\Gamma(G, \mathrm{Prob}(M))$

$\rightsquigarrow$ $\mathrm{Map}_\Gamma(G/P, \mathrm{Prob}(M))$.

(ii) $P \curvearrowright P/A$ is metrically ergodic (Mautner lemma).

$\Rightarrow G \curvearrowright G/A$ metrically ergodic

$\Rightarrow \Gamma \curvearrowright G/A$ metrically ergodic.

Thm: Any lcsc group G has a boundary.

(B-F, after Kaimanovich. Furstenberg–Poisson bndry of RW.)

Prop: If $G = G_1 \times \dots \times G_n$ then $B = B_1 \times \dots \times B_n$ is G-bnd

If $\Gamma < G$ lattice, then a G-boundary B is also a Γ-boundary.

Weyl Group

Defn Given a group Γ

Choose a Γ-boundary B

and let

$$W_{\Gamma,B} = \mathrm{Aut}_{\Gamma}^{\mathrm{meas}}(B \times B)$$

Ex. G - s-simple Lie group

$\Gamma < G$ lattice $\quad B = G/P$

$$W_{\Gamma,B} = N_G(A)/Z_G(A) = \mathrm{Weyl}_G$$

FOR $G = SL_n$ $\quad W = S_n$.

Ex $\Gamma < G = G_1 \times \dots \times G_n$ $\quad B = B_1 \times \dots \times B_n$

$$W_{\Gamma, B} \cong \mathbb{Z}/2 \times \dots \times \mathbb{Z}/2$$

$$B \times B = B_1 \times B_2 \times \dots \times B_n \times B_1 \times B_2 \times \dots \times B_n$$