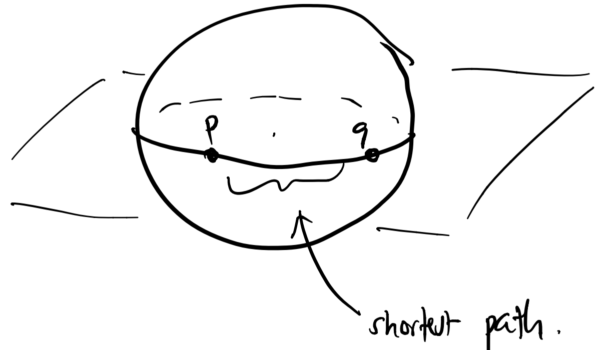
Lecture 1 Geometric structures on manifolds

The classical geometries

1) $\mathbb{R}^n$, with the Euclidean metric

2) $S^n \subset \mathbb{R}^{n+1}$, the unit sphere, equipped with the induced Riemannian metric.

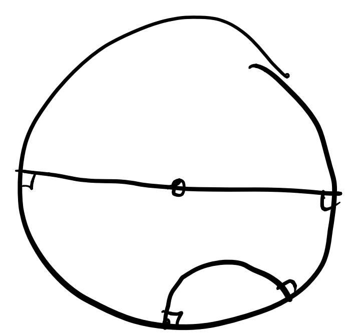
Geodesics, paths on S^n that locally minimize length, lie along great circles, the intersections of S^n with 2-plane through the origin.



3) $\mathbb{H}^n = \{ x \in \mathbb{R}^n \mid |x| < 1 \}$, equipped w/ the Riem metric

$$\langle , \rangle_{\mathbb{H}^n} = \left(\frac{2}{1-|x|^2} \right)^2 \langle , \rangle_{\mathbb{R}^n}.$$

So, hyp lengths are big near $\partial\mathbb{H}^n$, and geodesics are segments of circles (or lines) $\perp$ to $\partial\mathbb{H}^n$.



Fact If $X = \mathbb{R}^n, S^n$ or $\mathbb{H}^n$, $\mathrm{Isom}(X)$ acts transitively on X, and even on the set of ONBs for tangent spaces of X.

Def Space X is a Riem mfld and M is a top mfld. An X-structure on M is an atlas of charts

open

$$\begin{array}{ccc} \varphi: U_i & \longrightarrow & V_i \\ \cap & & \cap \\ M & & X \end{array}$$

s.t. transition maps are restrictions of isometries $X \longrightarrow X$. We call M an <u>X-manifold</u> when it has an X-structure. Note that the Riem metric on X pulls back under the charts to a well-defined Riem metric on M.

<u>Ex</u> S'pose $\Gamma < \mathrm{Isom}(X)$ acts on X prop disc, freely. Then

$$X \xrightarrow{\pi} \Gamma \backslash X =: M$$

is a covering map and M is an X-mnfld.

1) $T^n = \mathbb{Z}^n \backslash \mathbb{R}^n$ is a closed $\mathbb{R}^n$-mnfld.

2) $\mathbb{RP}^n = \mathbb{Z}_2 \backslash S^n$ is an <u>S^n-mnfld</u>. Also 3-dim lens space.
$\uparrow$ antipodal map

$$(x,y) \mapsto \left(e^{\frac{2\pi i}{p}} x,\ e^{2\pi i \frac{q}{p}} y\right)$$

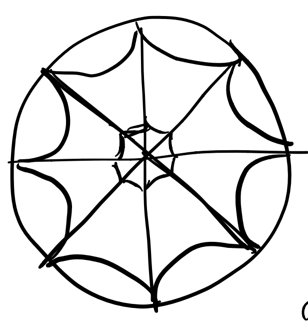
<u>Fact</u> If X is simply connected + complete (e.g. $\mathbb{R}^n, \mathbb{H}^n, S^n$) then every <u>complete</u> X-mnfld is of the form

$M = X/\Gamma$, $\Gamma < \mathrm{Isom}\, X$ prop disc, acting freely.

Rmk Need M complete, since e.g. the open disk in $\mathbb{R}^2$ is a $\mathbb{R}^2$-mnfld, but not a quotient

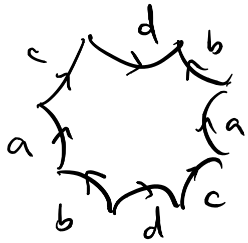
Ex (Hyp manifolds)

1)

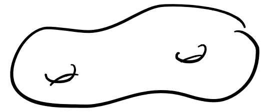


The interior angles of a regular hyp octagon vary from the Euclidean angles $\left(\frac{3\pi}{4}\right)$ to zero as edge length $\to \infty$.

So, there's one w/ $\frac{\pi}{4}$ interior angles. Gluing opposite sides, we get



$\cong$



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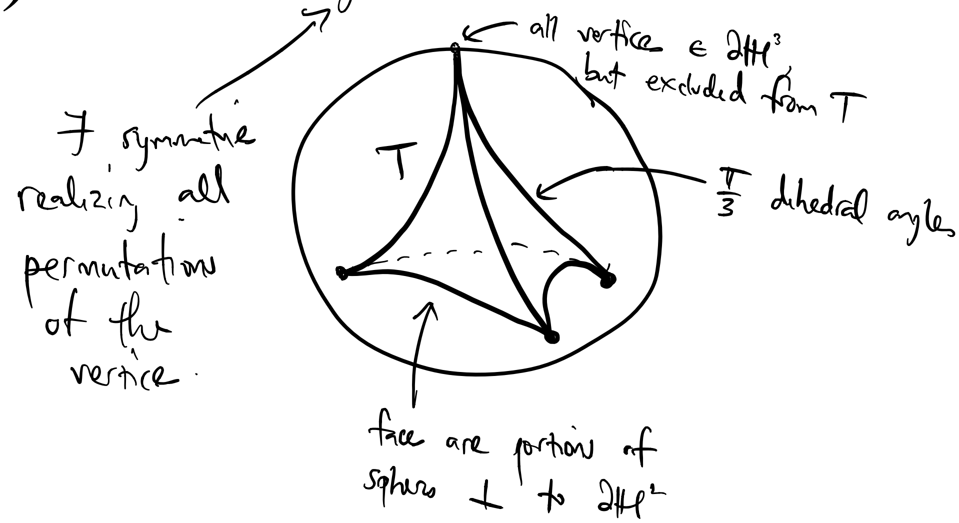
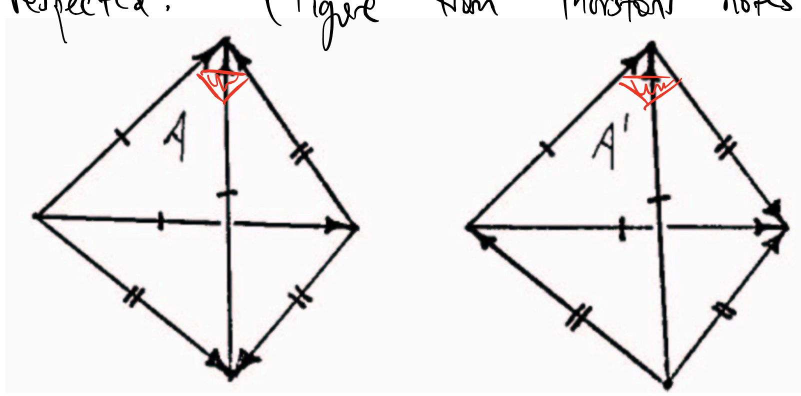
The vertices are all identified, giving a total angle of $2\pi = 8 \cdot \pi/4$ around the resulting vertex, which allows one to create a chart there into $\mathbb{H}^2$.

(Noncompact, w/ finite volume)

2) Take a regular ideal tetrahedron $T \subset \mathbb{H}^3$

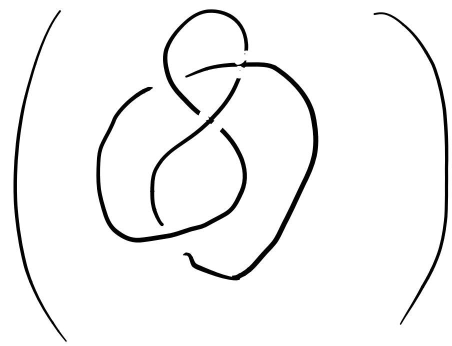


and glue two copies of T via the unique isometric face identifications s.t. all labels are respected. (Figure from Thurston's notes.)



Turns out

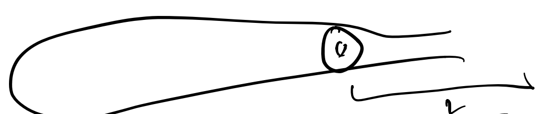
The result is homeo to the figure-8 knot complement, i.e.

$M = S^3 \setminus$ 

Six edges are glued at a time, so we get total dihedral angles of $2\pi = 6 \cdot \pi/3$, allowing us to construct charts into $\mathbb{H}^3$ (Note: there are no vertices of T, since they're at ∞.)

Note: M is not closed, but

$$\mathrm{vol}(M) = 2\,\mathrm{vol}(T) \approx 2.2.$$

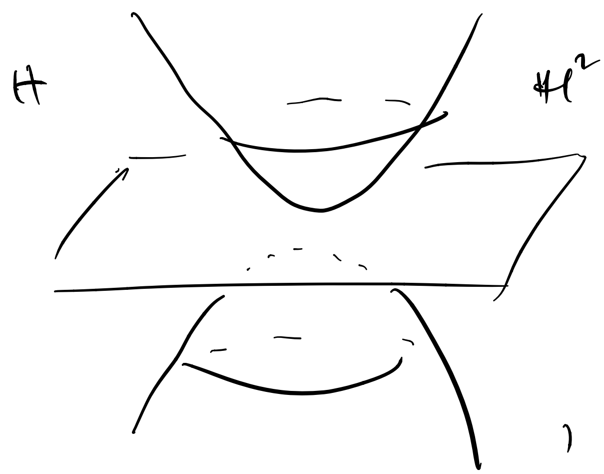


$\cong \qquad T^2 \times [0,\infty)$.

3) (Arithmetic constructions of hyp mfld w/ vol < ∞)

Rmk Set $\mathbb{R}^{n,1} = \mathbb{R}^{n+1}$ equipped w/ the quadratic form $q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$

The form q restricts to a norm on TH, where $H = \{q = -1\}$, and



$x \sim -x$ identifies the two sheets.

we have $\mathbb{H}^n \cong H/_{x\sim -x}$, when H is considered with the corr. Riem structure.

In this sense, $\mathbb{H}^n$ is like the sphere of radius $i = \sqrt{-1}$ in $\mathbb{R}^{n,1}$, and

$\mathrm{Isom}(\mathbb{H}^n) \cong SO(n,1) =$ linear autos of $\mathbb{R}^{n,1}$ preserving q and w/ det $= 1$.

where the det 1 condition is because $-I$ acts trivially on $\mathbb{H}^n \subset \mathbb{R}^{n+1}/_{x \sim -x}$.

Now let $\Gamma = SO(n,1) \cap SL_{n+1}\mathbb{Z}$.

Then Γ acts prop disc on $\mathbb{H}^n$, and the quotient has finite volume.

(Γ is a lattice in Isom $\mathbb{H}^n$! Compare w/ $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$.)

Γ doesn't act freely on $\mathbb{H}^n$, but by a lemma of Minkowski, the group

$$\Gamma(3) := \{ A \in \Gamma \mid A \equiv I \pmod 3 \}$$

is torsion free, and hence acts freely on $\mathbb{H}^n$, since no ∞-order isom of $\mathbb{H}^n$ that has a fixed pt acts prop disc.

Can also produce more "arithmetic" lattices (and hyperbolic n-mflds) from other representations $\text{Isom}\,\mathbb{H}^n \longrightarrow SL_m\mathbb{R}$, by intersecting w/ $SL_m\mathbb{Z}$.