# Rigidity Phenomena via Ergodic Theory. II

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**Prop:**

$\Gamma$ - a group. $(B,\nu)$ a $\Gamma$-boundary

$H \curvearrowright M$ a convergence action

$\rho : \Gamma \longrightarrow H$ with $\rho(\Gamma)$ non-elem.

Then

- $\exists!$ $\varphi : B \longrightarrow M$ measurable $\Gamma$-map

- $\exists 2$ $\Phi : B \times B \to M$ $\Gamma$-maps

$$\Phi_1(x,y) = \varphi(x)$$

$$\Phi_2(x,y) = \varphi(y).$$

**Recall:**

- $\Gamma \curvearrowright B$ amenable $\Rightarrow \exists$ $B \to \mathrm{Prob}(M)$
- $\Gamma \curvearrowright B \times B$ metrically ergodic $\Rightarrow$ ?

**Lemma:** (metric erg vs. properness).

> $\Gamma \curvearrowright (X,\mu)$ metrically ergodic.
> $Y$ loc. compact, $H \curvearrowright Y$ properly disc.
> $\rho : \Gamma \to H$ a hom.
> $f : X \to Y$ measurable $\Gamma$-map.
>
> Then: $\overline{\rho(\Gamma)}$ compact.

**Proof** $H \curvearrowright Y$ properly disc. implies

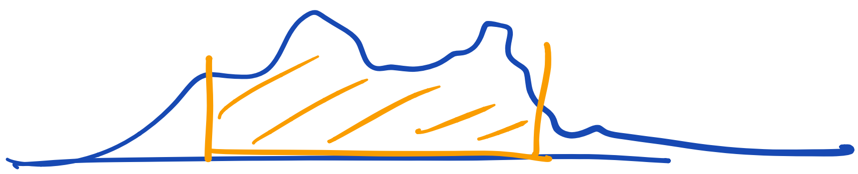
- Stabilizers $H_y = \mathrm{Stab}_H(y)$ compact
- Each orbit $H.y = H/H_y$ has $H$-inv. metric
- Space of orbits $Y/H$ is Hausdorff.

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[d] \\
 & Y/H
\end{tikzcd}
$$

Lemma:

$H \curvearrowright Y$ is properly disc

$\Rightarrow H \curvearrowright \mathrm{Prob}(Y)$ is properly discont.



$\Sigma \subset \mathrm{Prob}(M)$ is precompact

$\forall \varepsilon > 0 \;\exists\, \underset{\text{cpt}}{K \subset M},\quad \mu(K) > 1-\varepsilon \quad \forall \mu \in \Sigma$

How can this be used?

$$M \rightsquigarrow M^3 = M^{(3)} \sqcup \Delta$$

$$\Delta = \Delta_{12} \cup \Delta_{23} \cup \Delta_{31} \qquad \Delta_{ij} = \{(x_1, x_2, x_3) \mid x_i = x_j\}$$

$$\begin{tikzcd}
\mathrm{Prob}(M) \arrow[r] & \mathrm{Prob}(M^3) \arrow[dl, dashed] \arrow[dr, dashed] & H\text{-map} \\
\mathrm{Prob}(M^{(3)}) & & \mathrm{Prob}(\Delta)
\end{tikzcd}$$

$$\mu \longmapsto \mu * \mu * \mu$$

∘ ∘ ∘

<u>Back to</u> $\rho : \Gamma \to H \curvearrowright M$

$\Gamma \curvearrowright B$ is amenable

$\Rightarrow \exists\ \mu : B \to \mathrm{Prob}(M)$ $\quad x \longmapsto \mu_x$

$$\boxed{\mu_{\gamma . x} = \rho(\gamma)_* \mu_x}$$

Lemma $\Rightarrow$ $\quad B \times B \not\longrightarrow \mathrm{Prob}(M^{(3)})$

$\Rightarrow \mu_x * \mu_x * \mu_x (\Delta) = 1 \quad$ a.e. $x \in B$

$\Rightarrow \mu_x = \delta_{\varphi(x)}$.

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# Generalized Weyl Groups

$\Gamma$ a group

$B$ a $\Gamma$-boundary.

**Defn** $\quad W_{\Gamma,B} = \mathrm{Aut}_{\Gamma}^{\mathrm{meas}}(B \times B)$.

$W_{\Gamma,B} \supset \{\mathrm{Id}, \mathrm{flip}\}$

## Observation

$$W_{\Gamma,B} \curvearrowright \mathrm{Map}_{\Gamma}(B\times B, M) \doteq \{\Phi_1, \Phi_2\}$$

$$\Phi_i(x,y) = \varphi(x_i)$$

$$\boxed{\text{Higher Rank} \rightsquigarrow W_{\Gamma,B} \neq \mathbb{Z}/2}$$

**Case** $\quad G$ simple Lie grp. $\Gamma < G$ lattice

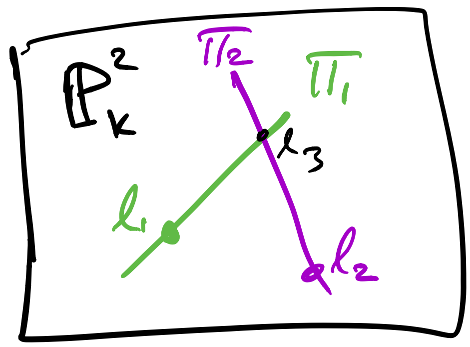
$B = G/P \qquad B \times B = G/A$

Take $B = G/P$ $\qquad B\times B = G/A$.

$$\mathrm{Aut}^{\mathrm{meas}}_{\Gamma}(B\times B) = \mathrm{Aut}^{\mathrm{meas}}_{\Gamma}(G/A)$$
$$\geq \mathrm{Aut}^{\mathrm{meas}}_{G}(G/A)$$
$$= N_G(A)/Z_G(A) = \mathrm{Weyl}_G$$

<u>Example</u>

$G = SL_3(k) \qquad \mathrm{Weyl}_G = S_3$




$B\times B \cong \{(l_1, l_2, l_3) \mid l_1 \oplus l_2 \oplus l_3\}$

$S_3$

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<u>Case</u> Lattices in a product

$\Gamma < G = G_1 \times \dots \times G_n \qquad n \geq 2.$

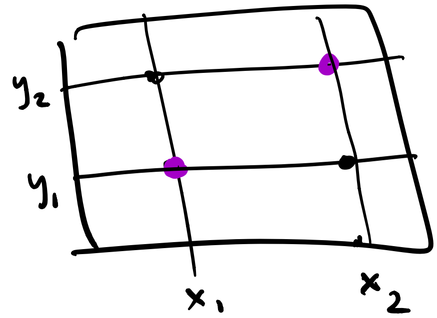
Take $B = B_1 \times B_2 \times \dots \times B_n$

$$W_{\Gamma, B} > (\mathbb{Z}/2)^n$$

$$B \times B = B_1 \times B_2 \times \cdots \times B_n \times B_1 \times B_2 \times \cdots \times B_n$$

$\tau_2$

$B \times B$




# Galois Correspondence

$$\mathcal{A} \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathcal{B}$$

POSets

$\alpha, \beta$ order-reversing such that

$$A \le \alpha(B) \iff \beta(A) \ge B$$

**Defn**

$\bar{A} := \alpha \circ \beta(A)$ $\quad \alpha \circ \beta : \mathcal{A} \to \mathcal{A}$

$\bar{B} := \beta \circ \alpha(B)$ $\quad \beta \circ \alpha : \mathcal{B} \to \mathcal{B}$

**Prop** $A \le \bar{A}$ $\quad B \le \bar{B}$ $\quad \bar{\bar{A}} = \bar{A}$ $\quad \bar{\bar{B}} = \bar{B}$

$$\begin{tikzcd} \bar{\mathcal{A}} \arrow[r, shift left, "\bar{\beta} = \bar{\alpha}^{-1}"] & \bar{\mathcal{B}} \arrow[l, shift left, "\bar{\alpha} = \bar{\beta}^{-1}"] \end{tikzcd}$$ anti-isom.

$\mathcal{A}$ poset of "closed" objects

poset of "closed" objects

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In Our Case $\quad j_1 : L^\infty(B) \hookrightarrow L^\infty(B \times B)$

$$\begin{pmatrix} \Gamma\text{-inv. subalgebras} \\ E < L^\infty(B, \nu) \end{pmatrix} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \begin{pmatrix} \text{subgroups} \\ V < W_{\Gamma, B} \end{pmatrix}$$

$$\alpha(E) = \mathrm{Fix}_{W_{\Gamma,B}}(j_1(E))$$

$$= \left\{ v \in W_{\Gamma,B} \;\middle|\; \begin{array}{l} f(p_1 \circ v(x,y)) = f(x) \\ f \in E \quad x, y \in B \end{array} \right\}$$

$$\beta(V) = \left\{ f \in L^\infty \;\middle|\; \begin{array}{l} f(p_1 \circ v(x,y)) = f(x) \\ \forall v \in V \quad x, y \in B \end{array} \right\}$$

**Prop:** For $G$ s-simple, $\Gamma < G$ lat

$B = G/P$ $\quad W_{\Gamma,B} = \mathrm{Weyl}_G$

Closed subgroups in $W_{\Gamma,B}$

$=$ special subgroups

(Weyl of Levi of parabolics).

Quotients of $G/P$ are $G/Q$

**Ex.**

$$\begin{tikzcd}[column sep=small]
 & \{1\} \arrow[dl, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\supset"] & \\
\langle (1\,2) \rangle \arrow[dr, phantom, sloped, "\subset"] & & \langle (2,3) \rangle \arrow[dl, phantom, sloped, "\supset"] \\
 & S_3 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & G/P = \{(\ell, \pi)\} \arrow[dl] \arrow[dr] & \\
G/Q_1 = \mathbb{P}^2 \arrow[dr] & & G/Q_2 = Gr_2 \arrow[dl] \\
 & \{*\} &
\end{tikzcd}$$