Geometrization of 3-mnflds, a la Thurston

Thm (Geometric Classification of surfaces)

Every closed surface S either an $\mathbb{R}^2$, $\mathbb{H}^2$ or S^2 structure; the type it admits is unique and determined by $\chi = \chi(S)$.

$\chi < 0$	$\chi = 0$	$\chi > 0$
$\mathbb{H}^2$	$\mathbb{R}^2$	S^2.

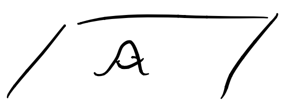
Pf idea Use classification of surfaces and explicitly construct metrics of the given forms, like we've already done for T^2, $\mathbb{RP}^2$ & .
Uniqueness is Gauss Bonnet. $\square$

Def A *model geometry* is a simply connected complete Riem mfld X with $\mathrm{Isom}\, X \curvearrowright X$

transitively, and where

1) $\exists$ a closed X-mfld
2) X is <u>maximally symmetric</u>.

Here, we write $X \preceq Y$ if $\exists$ a homeo $f: X \to Y$ with $f\,\mathrm{Isom}(X)\,f^{-1} \subseteq \mathrm{Isom}(Y)$, and $\sim$ if $=$. 2) means X is maximal wrt $\preceq$.

<u>Ex</u> 1)  $\preceq \quad \mathbb{R}^2$.

2) $(X, d) \sim (X, \alpha d)$, $\alpha > 0$.

<u>Thm</u> (Thurston) The 3-dim model geoms up to $\sim$ are

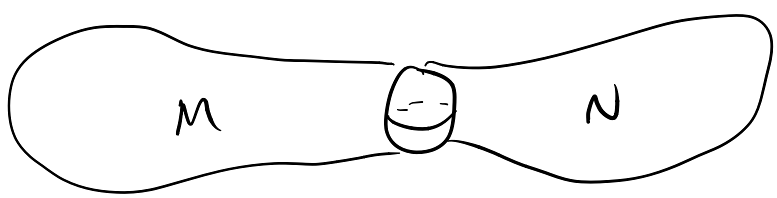
$$S^3,\ \mathbb{R}^3,\ \mathbb{H}^3,\ S^2 \times \mathbb{R},\ \mathbb{H}^2 \times \mathbb{R},\ \underbrace{\mathrm{Nil},\ \mathrm{Sol},\ \widetilde{SL_2\mathbb{R}}}_{\substack{\text{left inv't metrics} \\ \text{on Lie groups} \\ \text{homeo to } \mathbb{R}^3.}}$$

A manifold is <u>geometric</u> if it admits an X-structure, where X is a model geom..

Rmk (Not all 3-mflds are geometric)

M is irreducible if every embedded $S^2 \subset M$ bounds a ball. Every geometric M except $S^2 \times S^1$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$ is irreducible; those have $S^2 \times \mathbb{R}$ geom. (The other model geoms are homeo to S^3, $\mathbb{R}^3$, which are irreducible, $\Rightarrow$ w/ some work that their quotients are also.)

So, nontrivial connect sums are not geometric.



Also, let

$A =$  $\times S^1$, $\leftarrow$ compact, w/

$B = S^3 \setminus N(K)$, $\leftarrow$ $\partial = T^2$

where K is the figure 8 knot. Then

$$M = A \cup_{T^2} B$$

is not geometric.

The Geometrization Thm (Perelman '03, conj by Thurston)

Any irreducible, orientable closed 3-mfld contains a finite set of disjoint, embedded incompressible (π_1-injective) tori s.t. each complementary comp admits a complete fin vol X-structure for some model geom X.

Ex $M = ($  $\times S^1) \cup_{T^2} (S^3 \setminus N($  $))$
splits along T^2 int pieces with $\mathbb{H}^2 \times \mathbb{R}$ and hyperbolic geometry.

Rmk $\exists$ 19 model geoms in 4 dim, but probably no geometrization thm.

Application...

The Poincaré Conjecture If M is a closed simply connected 3-mfld, $M \cong S^3$.

Pf Splitting along spheres, we can reduce to the case that M is irreducible. Since

$\pi_1 M = 1$, M contains no π_1-injective tori. Hence geometrization $\Longrightarrow$ $M = \Gamma \backslash X$ is geometric. But $\Gamma \cong \pi_1 M \cong 1$, so $M \cong X$ is a model geometry. It's compact, so $M \cong S^3$. $\square$

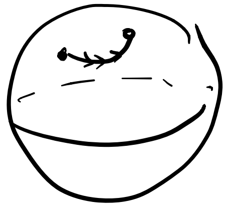
$\mathbb{Z}$

Def M is <u>atoroidal</u> if it does not have any embedded incompressible tori.

Fact Closed hyp 3-manifolds M are atoroidal.

Pf idea Writing $M = \Gamma \backslash \mathbb{H}^3$, an incompressible torus gives $\mathbb{Z}^2 \subset \Gamma$. . .

Classification of isoms of $\mathbb{H}^3$ $\Longrightarrow$ all elements of $\Gamma \setminus \mathrm{id}$ are translations along geodesic axes.



Commuting translations have the same axis, but $\mathbb{Z}^2$ can't act prop disc on $\mathbb{R}$. $\square$

Hyperbolization Thm Any atoroidal irreducible closed orientable 3-mfld M w/ infinite π_1 is hyperbolic.

The assumptions $\Rightarrow$ geometric. ∞ π_1 rules out S^3 geom. Irreducible rules out $S^2 \times \mathbb{R}$ geom and atoroidal rules out the rest.

Ex

S'pose S is a closed, orientable surface w/ genus $g \geq 2$ and $\varphi: S \to S$ is an orientation preserving homeo.

$$M_\varphi := \frac{S \times [0,1]}{(x,0) \sim (\varphi(x),1)}$$

is the mapping torus of φ.

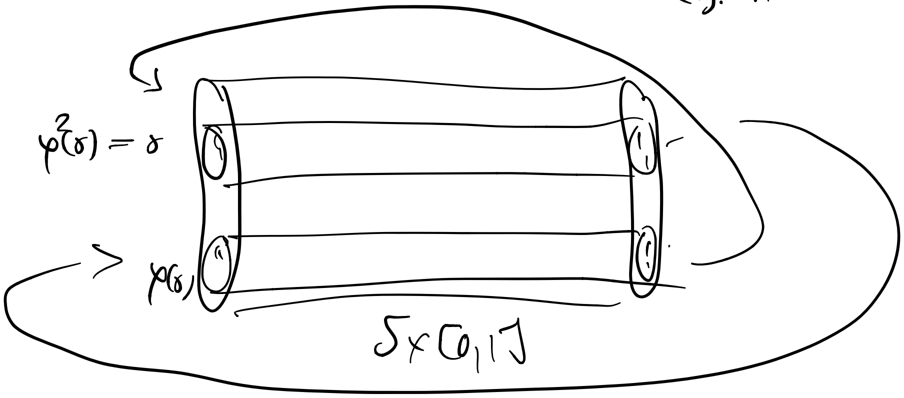
Thurston (1986) If φ is pseudo Anosov (pA) then M_φ is hyperbolic.

Here, φ is pA if for any s.c.c. $\gamma \subset S$, we have $\varphi^n(\gamma) \not\simeq \gamma$ $\forall n \in \mathbb{Z}$.
not homotopic

Note This precedes hyperbolization, but φ pseudo-Anosov $\iff$ M_φ atoroidal.

For instance, if $\varphi^n(\gamma) = \gamma$, we have an incompressible torus in M_φ:

e.g. $n=2$



Rmk Mapping tori are important first examples, & (Agol, + Wise '09 – '12) every closed hyp 3-mfld has a finite cover that's a mapping torus.

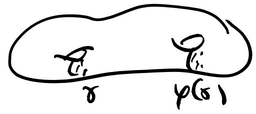
Ex If H is a handlebody w/ $\partial H = S$

 $H \subset \mathbb{R}^3$

and $\varphi: S \to S$ is a homeo, we can build a 3-manifold $N_\varphi = H \cup_\varphi H$.

This presentation is a Heegaard splitting of N_φ.

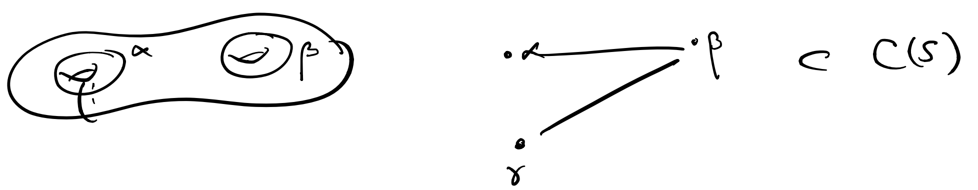
Note: N_φ is not hyperbolic if φ identifies the ∂'s of 2 disks in H.



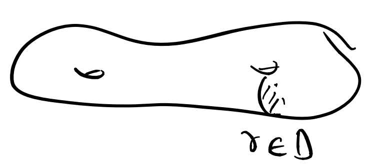
$\rightsquigarrow$

$S^2 \subset N_\varphi$.

Def The curve graph $C(S)$ has vertices that are homotopy classes of essential simple closed curves on S, and α, β are connected by an edge when they can be realized disjointly on S.



The disk set $D \subset C(S)$ is the set of curves that bound disks in H.



Thm (Hempel '01 + Geometrization)

N_φ is hyperbolic if $d_{C(S)}(D, \varphi(D)) \geq 3$.

<u>Pf idea</u> See how spheres and incompressible tori intersect the gluing surface. $\square$