Effective geometrization

Last time: A closed 3-mfld admits a hyp metric $\iff$ it's irreducible, atoroidal w/ ∞ π_1.

Mostow's Rigidity Thm: If a closed 3-mfld admits a hyp metric, the metric is unique up to isometry.

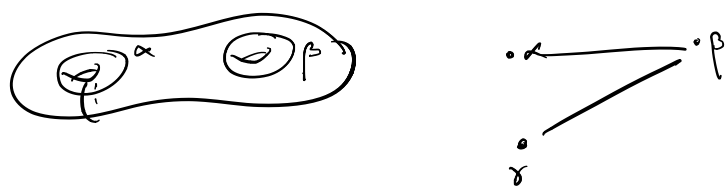
Q: S'pose M is hyperbolic. Can we read off geometric data (e.g. volume, diameter, etc...) from the topology of M?

"effective geometrization"

Ex: If $\varphi : S \to S$ is a pA homeo, how to understand the hyp metric on

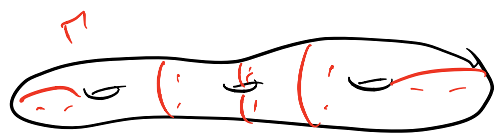
$$M_\varphi = S \times [0,1] \big/ (x,0) \sim (\varphi(x),1) \;?$$

Def The *curve graph* $C(S)$ has vertices that are homotopy classes of essential simple closed curves on S, and α, β are connected by an edge when they can be realized disjointly on S.

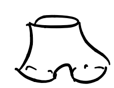


$\subset C(S)$.

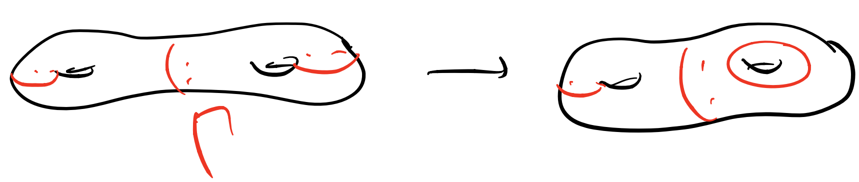
The *pants graph* $P(S)$ has vertices Γ that are pants decompositions of S, up to homotopy.



← set of curves dividing S into pants



Edges connect pants decompositions that differ by *elementary moves*, where a single curve $\gamma \in \Gamma$ is replaced by some γ' that intersects γ minimally.



Fact Make $C(S)$ and $P(S)$ metric space by setting all edge lengths $= 1$. Then both $C(S)$ and $P(S)$ are ∞-diam. (Masur-Minsky '99 for $C(S)$). Any homeo $\varphi : S \to S$ acts on both complexes via isometries

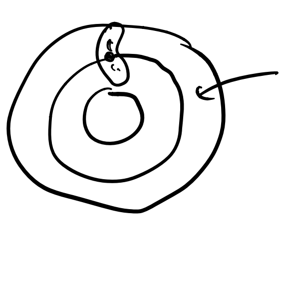
Defs If φ acts on a metric graph X, the translation distance $\tau_X(\varphi)$ is

$$\tau_X(\varphi) := \min_{\text{vertices } v} d(v, \varphi(v))$$

Thm (Brock 2003)

$\exists$ $K = K(\text{genus}(S))$, s.t. $\tau_{P(S)} \sim_K \text{vol}(M_\varphi)$!
where $A \sim_K B$ if $\frac{A}{K} \leq B \leq A$.

Rmks 1) $\tau_{C(S)}(\varphi) \sim_K$ the "electric circumference" of M_φ, see B-Souto '15.



(really, Brock–Bromberg, Bowditch, Minsky).

2) (Brock Canary Minsky 2012)

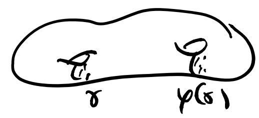
One can build a "combinatorial model" of $\hat{M}$ together w/ a K-bilipschitz homeomorphism $\mu: \hat{M} \longrightarrow M_\varphi$, using only the action of φ on $C(S)$ and certain projections of this action in the graphs $C(Y)$, $Y \subset S$. With patience, most pts of M_φ can then be quasi-encoded topologically

<u>Ex</u> If H is a handlebody w/ $\partial H = S$

$H \subset \mathbb{R}^3$

and $\varphi: S \to S$ is a homeo, we can build a 3-manifold $N_\varphi = \underbrace{H \cup_\varphi H.}_{\text{a \underline{Heegaard splitting}}}$

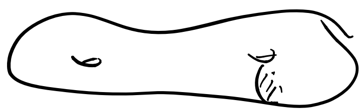
Ex: N_φ is not hyperbolic if φ identifies the ∂'s of 2 disks in H.



$\rightsquigarrow$

$S^2 \subset N_\varphi$.

The disk set $D \subset C(S)$ is the set of curves that bound disks in H.



Thm (Hempel '01 + Geometrization)

N_φ is hyperbolic if $d_{C(S)}(D, \varphi(D)) \geq 3$.

Pf idea See how spheres and incompressible tori intersect the gluing surface. $\square$

Many of the effective questions are harder, but in the works by various authors. (Brock–Souto, Lecuire–Namazi, etc.)

Rmk The mapping class group is the gp

$$MCG(S) = \{ f : S \to S \text{ homeo} \} / \text{homotopy}$$

It's finitely gen'd (Dehn - Lickorish) and if F is a finite gen set, we can create a random walk

$$\varphi_n = \gamma_1 \cdots \gamma_n ,$$

where the γ_i are chosen randomly from F. (Maher ~2010) showed that as $n \to \infty$, the prob that $M_{\varphi_n}, N_{\varphi_n}$ are hyperbolic $\to 1$. using the criteria above.

(This method of producing random 3-mfld was first considered by Dunfield-Thurston)

One ca show that the volumes of M_φ, N_φ increase linearly w/ n a.s. (see Brock, Maher + Viaggi '19)
Brock-Souto
Maher-Tiozzi

Q: How do more general top invts of a hyp 3-mfld constrain its geometry?

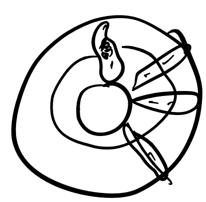
The rank of M is the min # of elements needed to generate $\pi_1 M$.

Question (McMullen)

Is the radius R_M of the largest embedded hyperbolic ball in M $\leq C = C(\text{rank } M)$?

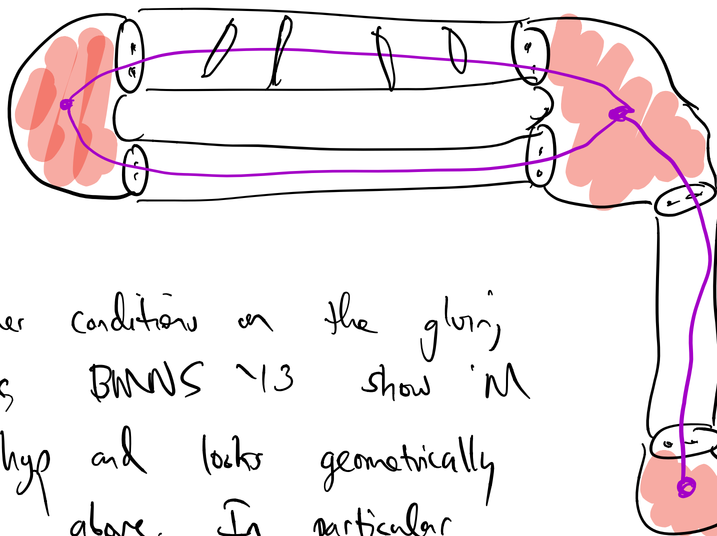
"if rank is bounded, is R_M bounded?"

Ex If genus(S) = g, then rank $M_\phi \leq 2g+1$.



Thurston $\Rightarrow$ M_ϕ is filled up with intrinsically hyperbolic surfaces ~ fibers, $\Rightarrow$ R_M is bounded.

More generally, can create mflds M w/ bounded rank by gluing a finite collection of fixed pieces together in the shape of a bounded complexity graph, with possibly very complicated gluing maps.



Under conditions on the gluing maps BMNS '13 show M is hyp and looks geometrically as above. In particular, no big embedded balls!

Thm (B-Souto '17) If rank M is bdd and inj(M) is bounded away from zero, then M looks as above.

Here, inj(M) = $\frac{1}{2}$ length of the shortest closed geodesic on M.

Cor $R_M \leq C = C(\text{rank } M, \text{inj}(M))$.