Virtual torsion in the homology of 3-manifolds

Michelle Chu
University of Illinois at Chicago
MSRI Research Member

based on joint work with Daniel Groves

Setting

M compact irreducible 3-manifold with empty or toroidal boundary, which is not a graph manifold

(i.e. at least 1 hyperbolic piece)

Goal

M virtually contains any prescribed torsion in homology

Motivation

$$H_1(M;\mathbb{Z}) = \mathbb{Z}^{b_1} \oplus \text{Torsion}$$

finite Abelian group

b_1: 1st Betti number

(a case of)

Lück Approximation thm:

Suppose $\cdots \longrightarrow M_i \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M$ is a cofinal tower of regular finite covers of M.

$\Gamma_i \trianglelefteq \cdots \trianglelefteq \Gamma_1 \trianglelefteq \Gamma$, $\quad \bigcap \Gamma_i = \{1\}$

Then

$$\lim_{i\to\infty} \frac{b_1(M_i)}{[\Gamma : \Gamma_i]} = 0$$

$b_1^{(2)}(M)$

L^2 1st Betti number

$=0$ (Lott-Lück)

Motivation

Virtually Special Thm (Agol, Wise, Przytycki–Wise)

$\Longrightarrow \quad vb_1(M) = \infty$

↑ virtual 1st Betti number

(i.e. $\forall k \in \mathbb{N} \ \exists$ fin. cover $\tilde{M} \longrightarrow M$ s.t. $b_1(\tilde{M}) \geq k$)

$\Longrightarrow$ we can find a tower of regular covers $\{M_i\}$

with $b_1(M_i) \longrightarrow \infty$ but very slowly

w.r.t. degree of the covers

Motivation

Conjecture (Bergeron-Venkatesh, Lück, Le):

There exists a cofinal tower of regular finite covers $\{M_i\}$ s.t

$$\lim_{i\to\infty} \frac{\log |\mathrm{Tor}(H_1(M_i;\mathbb{Z}))|}{[\Gamma : \Gamma_i]} = \frac{\mathrm{vol}(M)}{6\pi}$$

vol (M) = sum of volumes of the hyperbolic pieces

Le : proved $\leq$

Motivation

Question:

Does there exist $\tilde{M} \longrightarrow M$ finite cover

s.t. $\mathrm{Tor}(H_1(\tilde{M};\mathbb{Z})) \neq 0$?

Answer:

Yes — Hongbin Sun closed hyperbolic 2015

2017 Liu regular cover

Friedl-Herrmann Torsion group as large as you want

Motivation

Thm (Sun 2015):

N closed hyperbolic 3-manifold

A finite abelian group

Then $\exists$ finite cover $\tilde{N} \longrightarrow N$ s.t. A is a direct summand in $H_1(\tilde{N};\mathbb{Z})$.

Question:

Does this hold more generally?

Theorem (C-Groves):

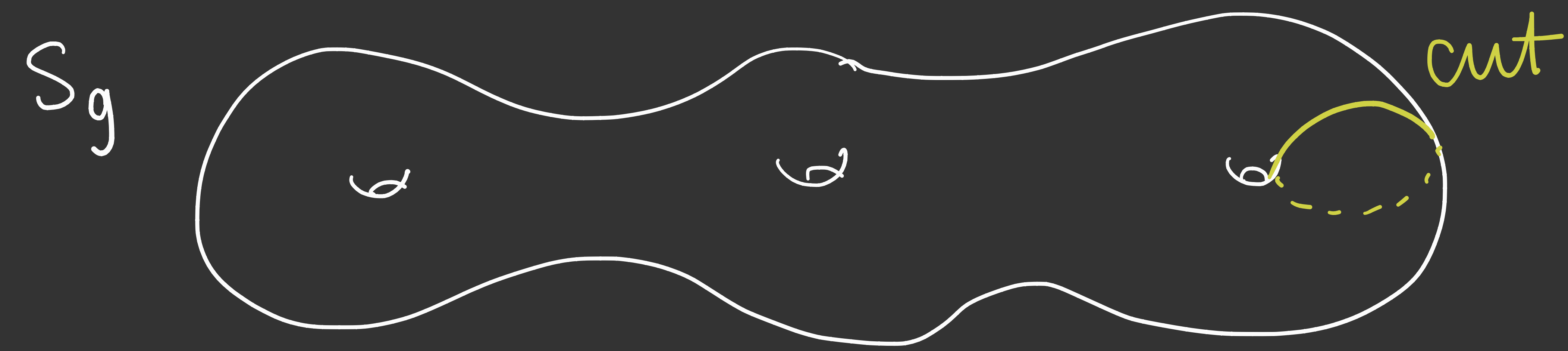
M irreducible 3-manifold with empty or toroidal boundary with at least 1 hyperbolic piece

A finite abelian group

Then $\exists\ \tilde{M} \to M$ finite cover s.t. A is a direct summand in $H_1(\tilde{M};\mathbb{Z})$.

* The key case is finite volume hyperbolic

How to construct torsion in homology?



Exercise: $H_1(X_n; \mathbb{Z}) = \mathbb{Z}^{2g-1} \oplus \mathbb{Z}/n\mathbb{Z}$

How to construct torsion in homology?

Suppose Y retracts to X

i.e. $X \xrightarrow{i} Y$ and $r: Y \longrightarrow X$

s.t. $r \circ i = \mathrm{Id}_X$

Exercise: Then $H_k(Y) = H_k(X) \oplus \mathrm{Kernel}(r_*)$

Sun's Strategy for closed hyperbolic:

* case $A = \mathbb{Z}/n\mathbb{Z}$

- Start with a Kahn-Markovic surface S
  ↳ with a particular good cuff C (non-separating) that goes along some geodesic γ n-times + more

- Cut along C and take a quotient by $\frac{2\pi}{n}$-rotation to get X_n

Sun's Strategy for closed hyperbolic:

- Do all this in such a way that you can guarantee $X_n \hookrightarrow M$ is π_1-injective and $\pi_1(X_n)$ is a quasi-convex subgroup of $\pi_1(M)$

- By the virtually special thm and virtual retractions $\exists\ \tilde{M} \longrightarrow M$ finite cover s.t. $\pi_1(\tilde{M})$ retracts to $\pi_1(X_n)$

finite volume

Sun's Strategy for ~~closed~~ hyperbolic:

$*$ case $A = \mathbb{Z}/n\mathbb{Z}$

Kahn-Wright

- Start with a ~~Kahn-Markovic~~ surface S
  ↳ with a particular good cuff C (non-separating) that goes along some geodesic γ n-times + more and stays far from the cusps + more

- Cut along C and take a quotient by $\frac{2\pi}{n}$-rotation to get X_n

finite volume

Sun's Strategy for ~~closed~~ hyperbolic:

- Do all this in such a way that you can guarantee $X_n \hookrightarrow M$ is π_1-injective and $\pi_1(X_n)$ is a relatively quasi-convex subgroup of $\pi_1(M)$

- By the virtually special thm and virtual retractions $\exists\ \tilde{M} \longrightarrow M$ finite cover s.t. $\pi_1(\tilde{M})$ retracts to $\pi_1(X_n)$

relative setting

Thank you