Tools for Counting Quaternion Algebras

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I. Geometric Motivation

Let M, N be Compact Riemannian manifolds.

3 Notions of "equivalence":

① M & N are Commensurable if they have a common finite degree Covering space.

② M & N are isometric if $\exists$ an isometry between them.

③ M & N are isospectral if $\mathcal{E}(M) = \mathcal{E}(N)$ & length isospectral if $LS(M) = LS(N)$

multiset of eigenvalues of the Laplacian of M

↳ multiset of lengths of closed geodesics on M

"Inverse" questions: to what extent do the spectra of M determine its geometry & topology?

Ex

① If $LS(M) = LS(N)$, are M & N isometric? No ← Can't hear the shape of a drum

② If $LS(M) = LS(N)$, are M & N Commensurable?

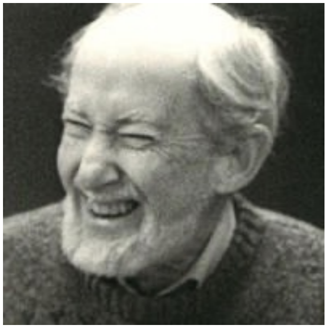
we'll focus on this one

↳ Arithmetic case: Yes

↳ Non-arithmetic case: ???

Still open

1



Leon Green (1960) asked if the spectrum of M determines its isometry class.

The spectrum of M is essentially the collection of frequencies produced by a drumhead shaped like M.

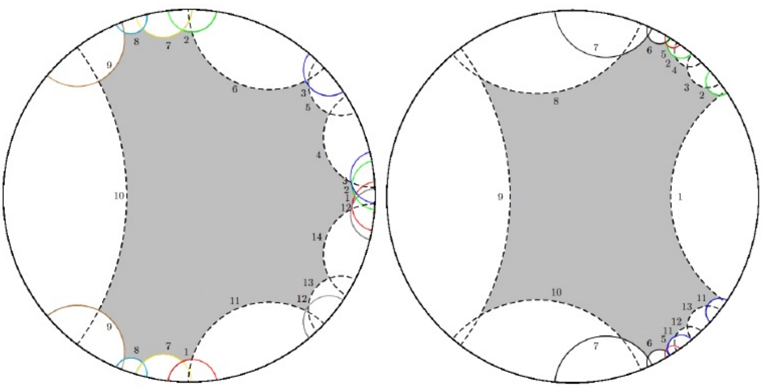
Mark Kac (1966) popularized this question for planar domains:

Can you hear the shape of a drum?

① Isospectral $\not\Rightarrow$ Isometric

Theorem (Vigneras, 1980)

There exist isospectral non-isometric hyperbolic 2- and 3-manifolds.



A pair of isospectral but non-isometric hyperbolic 2-orbifolds (due to B. Linowitz and J. Voight).

① Isospectral $\not\Rightarrow$ Isometric

Theorem (Gordon, Webb, Wolpert, 1992)

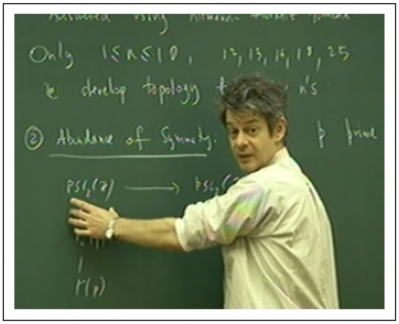
One cannot hear the shape of a drum.

$\Updownarrow$

$\exists$ isospectral nonisometric planar domains

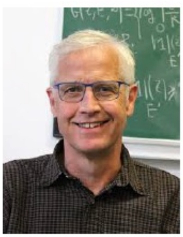
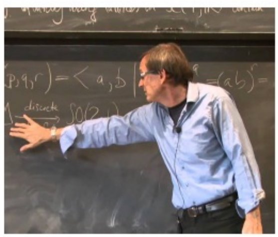
②

If $LS(M) = LS(N)$, are M & N commensurable?



Theorem (Reid, 1992)

If M is an arithmetic, hyperbolic surface and $\mathrm{LS}(M) = \mathrm{LS}(N)$ then M and N are commensurable.

Theorem (Chinburg, Hamilton, Long, Reid, 2008)

If M and N are arithmetic hyperbolic 3-manifolds and $\mathrm{LS}(M) = \mathrm{LS}(N)$ then M and N are commensurable.

on the other hand...

Theorem (Futer and Millichap, 2016)

For every sufficiently large $n > 0$ there exists a pair of non-isometric finite-volume hyperbolic 3-manifolds $\{M, N\}$ such that:

1. $\mathrm{vol}(M) = \mathrm{vol}(N)$.
2. *The (complex) length spectra of M and N agree up to length n.*
3. *M and N have at least e^n/n closed geodesics up to length n.*
4. *M and N are not commensurable.*

* Non-arithmetic

Our motivating Question:

If two arithmetic hyperbolic 2- or 3-manifolds have many overlapping geodesic lengths, are they necessarily commensurable?

↳ No. "Lots" of p-w non-commensurable.

Key Idea:

$$\left\{\begin{array}{l}\text{Maximal subfields}\\ \text{of quaternion algebras}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{lengths of}\\ \text{geodesics on}\\ \text{arithmetic hyperbolic}\\ \text{2- \& 3-manifolds}\end{array}\right\}$$

number theory

geometry

How we Use This:

① Translate geometric question into a # theory question via this correspondence

② Apply analytic # theory techniques

③ Translate back to geometry - get quantitative result

Much of our work boils down (essentially) to counting quaternion algebras w/ bdd discriminant (subject to additional, geometrically-motivated constraints).

Plan

Talk #1: Intro to Analytic NT tools for counting quaternion algebras

↳ no knowledge of quaternion algebras or geometry needed!

Talk #2: Introduce quaternion algebras, show how to count them, derive geometric consequences

Talk #3: Application of gaps between primes to spectral geometry!

II. Arithmetic Functions & Dirichlet Series

Def An *arithmetic function* is any function $f : \mathbb{N} \to \mathbb{C}$.

Examples:

① $\tau(n) :=$ # of positive divisors of n, i.e.,

$$\tau(n) = \sum_{d \mid n} 1$$

Ex $\tau(6) = 4$

② $\varphi(n) := \#\{k \in [1, n] : \gcd(k, n) = 1\}$

Ex $\varphi(6) = 2$

③ $\omega(n) =$ # of distinct prime factors of n,

i.e., $$\omega(n) = \sum_{p \mid n} 1$$

Ex $\omega(6) = 2$

④ $\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } n \text{ is } \square\text{-free} \\ 0 & \text{otw} \end{cases}$

⑤ $\mathbb{1}(n) = 1 \quad \forall n$

* To each arithmetic function $f: \mathbb{N} \to \mathbb{C}$, we can associate a generating function.

Def Given $f: \mathbb{N} \to \mathbb{C}$, define the Dirichlet Series of f to be

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Ex If $f = \mathbb{1}$,

$$D_{\mathbb{1}}(s) = \sum_{n=1}^{\infty} \frac{\mathbb{1}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

Riemann zeta function

III. Multiplying Dirichlet Series

* There are various ways of multiplying Dirichlet series (depending on how you collect the terms).

Ex The Dirichlet product of $\sum_{n=1}^{\infty} a_n$ and $\sum_{m=1}^{\infty} b_m$ collects all terms $a_n b_m$ for which nm is constant, i.e., $n \cdot m = N$. So, if $\sum_{n=1}^{\infty} a_n$ & $\sum_{m=1}^{\infty} b_m$ are abs. conv.,

$$\sum_{N=1}^{\infty} \left(\sum_{nm=N} a_n b_m \right) = \sum_{n=1}^{\infty} a_n \sum_{m=1}^{\infty} b_m$$

new arithmetic function

$\therefore$ Product of Dirichlet Series is also a Dirichlet Series

This motivates:

<u>Def</u> If f & g are arithmetic then the <u>Dirichlet Convolution</u> $f*g$ is defined

$$f*g(n) = \sum_{ab=n} f(a)g(b)$$

$$= \sum_{d|n} f(d)g(n/d) \quad \forall n \in \mathbb{N}$$

$$\left(\text{equiv: } \sum_{d|n} f(n/d)g(d)\right)$$

Thus:

$$\boxed{D_f(s)D_g(s) = D_{f*g}(s)}$$

$\forall s \in \mathbb{C}$ where $D_f(s)$ & $D_g(s)$ converge absolutely.

__Example__ $\forall n \geq 1$:

The function that is 1 on every input

$$\mathbb{1} * \mathbb{1}(n) = \sum_{d \mid n} \mathbb{1}(d)\,\mathbb{1}(n/d)$$

$$= \sum_{d \mid n} 1$$

$$= \tau(n)$$

Same function from earlier

__Example__

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = D_\tau(s)$$

$$= D_{\mathbb{1} * \mathbb{1}}(s)$$

$$= D_{\mathbb{1}}(s)\, D_{\mathbb{1}}(s)$$

$$= (\zeta(s))^2$$

Riemann Zeta

Def An arithmetic function f is **multiplicative** if

$$\boxed{f(ab) = f(a)f(b) \quad \forall a,b \in \mathbb{N} \text{ w/ } \gcd(a,b)=1.}$$

* when f is multiplicative, then

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \underbrace{\prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right)}$$

This is called the Euler product expansion

IV. Using Dirichlet Series in Counting Problems

General Idea:

① Construct a Dirichlet series whose coeffs count a quantity we are interested in.

Classic Example

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad a_n = \begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otw} \end{cases}$$

Coeffs of Dirichlet Series Count Prime numbers. Used to prove the prime # Theorem.

② Check that the Dirichlet Series Satisfies certain "nice" analytic properties so that we can compute the limit of the partial sums of the coeffs (gives an <u>asymptotic</u>)

<u>Ex</u>

$$\underbrace{\sum_{n \le x} a_n}_{\text{Partial Sum of Coeffs of Dirichlet series above}} = \pi(x)$$

# of primes $\le x$

* Check analytic Conditions of $\sum_n \frac{a_n}{n^s}$ *

more about this in a moment

Applying a Tauberian Thm gives:

Prime # Thm: $\pi(x) \sim \frac{x}{\log x}$
(PNT)

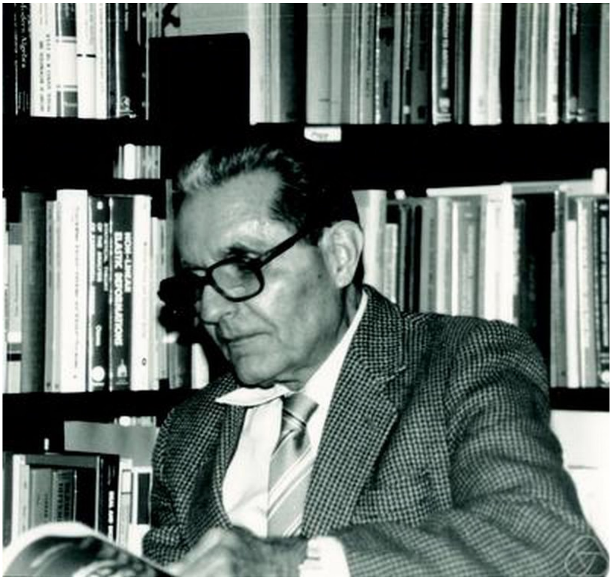
$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$

Tauberian Thms roughly say: if our Dirichlet Series satisfies "nice" analytic properties then we can take the limit inside of the sum (or integral).

Examples

1. Wiener - Ikehara → Commonly used for PNT
2. Newman's Analytic → used for a different formulation of PNT (w/ integrals)
3. Hardy - Littlewood → used to show $\sum_n \frac{\mu(n)}{n}$ converges
4. Delange → we use this one to get an asymptotic for the quaternion algebras we are interested in

we'll use this one...



Theorem (Delange's Tauberian Theorem)

Let $G(s) = \sum \frac{a_N}{N^s}$ *be a Dirichlet series satisfying:*

1. $a_N \geq 0$ *for all* N *and* $G(s)$ *converges for* $\Re e(s) > \rho$.
2. $G(s)$ *can be continued to an analytic function in the closed half-plane* $\Re e(s) \geq \rho$ *except possibly for a singularity at* $s = \rho$.
3. *There is an open neighborhood of* ρ *and functions* $A(s), B(s)$ *analytic at* $s = \rho$ *with* $G(s) = A(s)/(s-\rho)^{\beta} + B(s)$ *at every point in this neighborhood having* $\Re e(s) > \rho$.

Then as $x \to \infty$ *we have*

$$\sum_{N \leq x} a_N = \left(\frac{A(\rho)}{\rho \Gamma(\beta)} + o(1) \right) x^{\rho} \log(x)^{\beta - 1}.$$