Random walks on weakly hyperbolic groups

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Random and Arithmetic Structures in Topology
MSRI - Fall 2020

Random walks on weakly hyperbolic groups - Summary

- **Lecture 1** (Aug 31, 10.30): Introduction to random walks on groups

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Random walks, WPD actions, and the Cremona group

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Answer. It depends on the topography (geometry) of the city.

Recurrent random walks

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Recurrence

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Exercise. Prove the Lemma.

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$\therefore$ our RW is recurrent.

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Exercise. Prove Polya's theorem for $d = 3$. Moreover, for the simple random walk on $\mathbb{Z}^d$, show that $p^{2n}(0,0) \approx n^{-\frac{d}{2}}$.

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(do we know $\lim_{n\to\infty} \frac{d_n}{n}$ exist?)

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Example: the group of reals acting on itself by translations: $X = \mathbb{R}$, $G = \mathbb{R}$ and the action $\rho : \mathbb{R} \to \text{Isom}(\mathbb{R})$ is given by $\rho(t) : x \mapsto x + t$.

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and define the sample space as the space $(\Omega, \mathbb{P})$ where $\Omega = G^{\mathbb{N}}$ and $\mathbb{P} = \Phi_\star \mu^{\mathbb{N}}$ is the pushforward.

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Moreover, we define the word metric or word distance between $g, h \in G$ as

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Let (X, d) be a geodesic, metric space, and let $x_0 \in X$ be a basepoint.

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The geodesic metric space X is δ-hyperbolic if $\exists \delta > 0$ such that geodesic triangles are δ-thin.

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Recall a space is proper if metric balls $\{z \in X \ : \ d(x, z) \leq R\}$ are compact.

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