Quantitative Questions in Spectral Geometry

Lola Thompson

Universiteit Utrecht

Recall from yesterday:

Our motivating Question:

If two arithmetic hyperbolic 2- or 3-manifolds have many overlapping geodesic lengths, are they necessarily commensurable?

↳ No. "Lots" of p-w non-commensurable.

Key Idea:

$$\left\{ \begin{array}{l} \text{Maximal subfields} \\ \text{of quaternion algebras} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{lengths of} \\ \text{geodesics on} \\ \text{arithmetic hyperbolic} \\ \text{2- \& 3-manifolds} \end{array} \right\}$$

↑ number theory

↑ geometry

Today's plan:

1. Introduce quaternion algs.
2. Show how to construct a Dirichlet series whose coeffs. count quaternion algebras
3. Apply a Tauberian Thm to obtain an asymptotic for the count of quaternion algs w/ $|\text{disc}| \leq x$ (w/ certain geometrically-motivated conditions).
4. Derive geometric consequences!

I. Quaternion algebras & orders

Theorem (Hamilton, 1843)

The $\mathbb{R}$-algebra $\mathbb{H}$ with basis $\{1, i, j, ij\}$ and defining relations

$$i^2 = -1 \qquad j^2 = -1 \qquad ij = -ji$$

is a four-dimensional division algebra.

Brougham Bridge:

Notation: write $(-1, -1, \mathbb{R})$ instead of $\mathbb{H}$.

Can replace w/ other units in $\mathbb{R}$

Can be replaced w/ other fields of char 0

Ex $(1, 1, \mathbb{R})$ has $i^2 = j^2 = 1$ &

$ij = -ji$.

observe:

$$(1, 1, \mathbb{R}) \cong M_2(\mathbb{R})$$

$$i \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In general:

$$(a,b,\mathbb{R}) \cong \begin{cases} \mathbb{H} & \text{if } a,b<0 \\ M_2(\mathbb{R}) & \text{otw} \end{cases}$$

Thus, $(a,b,\mathbb{R})$ is either a division algebra or isomorphic to $M_2(\mathbb{R})$.

$\mathbb{H}$

* If we replace $\mathbb{R}$ w/ other fields:

Theorem (Wedderburn)

For any field F, if the F-algebra (a,b,F) is not a division algebra then $(a,b,F) \cong \mathrm{M}_2(F)$.

In general, there won't be a unique division alg /F.

Extension of Scalars

Let K be a field, K'/K a field ext.

If $B = (a, b, K)$ is a quaternion alg $/K$, then

$$B \otimes_K K' = (a, b, K')$$

is a quaternion alg $/K'$.

Important for arithmetic applications

Def If B is a quaternion alg $/K$, let

$$B_{\mathfrak{p}} := B \otimes_K K_{\mathfrak{p}}.$$

we say B is ramified at $\mathfrak{p}$ if $B_{\mathfrak{p}}$ is the unique division alg $/K_{\mathfrak{p}}$. Otw, B splits in $\mathfrak{p}$.

Def Let Ram(B) denote the (finite) set of primes at which B is ramified. The discriminant of B is the ideal defined by

$$\Delta(B) := \prod_{\mathfrak{p} \in \mathrm{Ram}(B)} \mathfrak{p}.$$

Def Let B be a quaternion alg / K. The reduced norm of B is the composite map

number field

$$B \hookrightarrow M_2(\mathbb{C}) \xrightarrow{\det} \mathbb{C}$$

Let K be a number field w/ ring of integers $\mathcal{O}_K$.

Def An *order* of a K-alg. is a subring which is also a f.g. $\mathcal{O}_K$-module containing an K-basis of the algebra.

Def An order is *maximal* if it is not properly contained in any other order.

Ex $M_2(\mathbb{Z})$ is a maximal order of $M_2(\mathbb{Q})$

Ex $\mathcal{O}_K$ is a maximal order of K.

How arithmetic Surfaces arise from quaternion algebras

Elementary results from geometric group theory:

- $Isom^+(\mathbf{H}^2) \cong PSL_2(\mathbb{R})$.
- Every orientable hyperbolic 2-manifold is of the form $\mathbf{H}^2/\Gamma$ for some discrete subgroup Γ of $PSL_2(\mathbb{R})$.

We want to generalize the following construction of $PSL_2(\mathbb{Z})$:

$$M_2(\mathbb{Q}) \supset M_2(\mathbb{Z}) \to SL_2(\mathbb{Z}) \to PSL_2(\mathbb{Z})$$

Replace w/ quaternion alg B/k in which a unique real prime splits

replace w/ maximal quaternion order $\mathcal{O}$

Consider the embedding $\rho : B \to M_2(\mathbb{R})$.

Restricting ρ to $\mathcal{O}^1$ ← elts of $\mathcal{O}$ w/ reduced norm 1

& Projecting onto $PSL_2(\mathbb{R})$

gives an embedding:

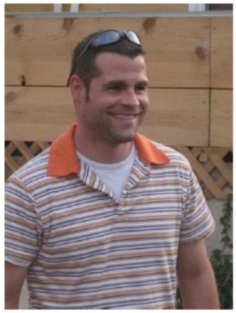
$$\bar{\rho} : \mathcal{O}^1 \to PSL_2(\mathbb{R}).$$

Notice:

* $\bar{\rho}(\mathcal{O}^1)$ is a discrete s.g. of isometries w/ finite covolume
* If B is a division alg, then $\bar{\rho}(\mathcal{O}^1)$ is cocompact.
* If $\bar{\rho}(\mathcal{O}^1)$ is torsion-free then $\mathbb{H}^2/\bar{\rho}(\mathcal{O}^1)$ is a hyperbolic 2-manifold.

hyperbolic surfaces commensurable w/ things of this form are arithmetic

II. Counting Quaternion Algebras w/ Prescribed Embeddings

Theorem (Linowitz, McReynolds, Pollack, T., 2018)

Fix a number field k, and fix quadratic extensions $L_1, L_2, \ldots, L_r$ of k. Let L be the compositum of the L_i, and suppose that $[L : k] = 2^r$. The number of quaternion algebras over k with discriminant having norm less than x and which admit embeddings of all of the L_i is

$$\sim \delta \cdot x^{1/2}/(\log x)^{1-\frac{1}{2^r}},$$

as $x \to \infty$. Here δ is a positive constant depending only on the L_i and k.

without this condition, it's possible that no quaternion alg. will admit embeddings of all the L_i's.

* We sketch a proof of the special case where $r=1$, i.e.,

$$\#\left\{ \begin{array}{l} B/K \text{ w/ } |\operatorname{disc} B| < x : \\ B \text{ admits an embedding} \\ \text{of } L/K \end{array} \right\} \sim \underset{\substack{\text{depends on} \\ L \,\&\, K}}{C} \frac{x^{1/2}}{\sqrt{\log x}}$$

as $x \to \infty$

Black Box

Useful results from algebraic number theory:

(1) $p \mid \Delta \iff p$ ramifies in $B \iff B \otimes K_p$ is a division algebra.

(2) The total number of primes that ramify in B is even.

(3) For $\{p_1, \ldots, p_\ell\}$ in K with even cardinality, there exists a unique quaternion algebra B/K that ramifies at those primes (and hence has $\Delta = p_1 \cdots p_\ell$).

(4) L embeds into $B \iff$ no prime of K that divides Δ splits in L/K (hence, Δ is squarefree).

Sketch of Pf $(r=1)$

Let $\mathcal{P} = \{p_i \mid \Delta : p_i \text{ finite}\}$
$\quad\;\; \mathcal{Q} = \{q_j \mid \Delta : q_j \text{ infinite}\}$

By ①, these ramify in B.

From Black Box:

$(*)$ $\displaystyle \Delta = \prod_{\substack{1 \le i \le n \\ p_i \in \mathcal{P}}} p_i \prod_{\substack{1 \le j \le m \\ q_j \in \mathcal{Q}}} q_j$ where:

From ③

① All p_i's, q_i's are distinct (i.e., Δ $\square$-free)

From ④

② $n+m$ is <u>even</u>

From ②

Thus, our task is to count:

$$\#\{|\Delta| \le X : \Delta \text{ satisfies } (*)\}$$

Let $F_1(s) = \prod_{\substack{p \text{ finite or infinite} \\ p \text{ does not split in } L}} \left(1 + \frac{1}{|p|^s}\right)$

Can rewrite $F_1(s)$ as:

$$\sum_{\substack{\Delta \text{ as in } (*) \\ \text{except w/ out} \\ \text{Condition that} \\ n+m \text{ is even}}} \frac{1}{|\Delta|^s}$$

↳ To make up for this:

Let $F_2(s) = \prod_{\substack{p \text{ finite or infinite} \\ p \text{ does not split in } L}} \left(1 - \frac{1}{|p|^s}\right)$

Let $F(s) := \frac{1}{2}\left(F_1(s) + F_2(s)\right)$

Then $F(s)$ has $2 \mid (m+n)$ $\forall \Delta$, so

$$F(s) = \sum_{\Delta \text{ as in } (*)} \frac{1}{|\Delta|^s}$$

Upshot: we can estimate

$$\#\left\{\begin{array}{l} B/k \text{ w/ } |\Delta| \le x : \\ B \text{ admits an} \\ \text{embedding of } L/k \end{array}\right\}$$ by taking

$\frac{1}{2}$(partial sums of coeffs of $F_1(s)$) +

$\frac{1}{2}$(partial sums of coeffs of $F_2(s)$)

How to take partial sums of coeffs:

* First, reformulate $F_1(s)$ so that we only need to sum over finite primes

Note: If p is an infinite prime, then

$$1 + \frac{1}{|p|^s} = 1 + \frac{1}{1^s} = 2.$$

So $F_1(s) = 2^{r} \cdot \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ ← non-negative mult. function

# of infinite primes that do not split in K

$$f(n) = \begin{cases} 1 & \text{if } n \text{ satisfies } \circledast \text{ w/out even \# of prime factors condition} \\ 0 & \text{otw} \end{cases}$$

* Similar for $F_2(s)$:

$$F_2(s) = \left(\text{constant factor coming from infinite primes}\right) \cdot \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

* Use Wirsing's Thm + partial summation to show:

$$\sum_{n \leq x} b(n) = o\left(\sum_{n \leq x} a(n)\right)$$

What Remains:

① Check analytic Conditions

② Apply Delange's Tauberian Thm

Theorem (Delange's Tauberian Theorem)

Let $G(s) = \sum \frac{a_N}{N^s}$ be a Dirichlet series satisfying:

1. $a_N \geq 0$ *for all* N *and* $G(s)$ *converges for* $\Re e(s) > \rho$.
2. $G(s)$ *can be continued to an analytic function in the closed half-plane* $\Re e(s) \geq \rho$ *except possibly for a singularity at* $s = \rho$.
3. *There is an open neighborhood of* ρ *and functions* $A(s), B(s)$ *analytic at* $s = \rho$ *with* $G(s) = A(s)/(s-\rho)^\beta + B(s)$ *at every point in this neighborhood having* $\Re e(s) > \rho$.

Then as $x \to \infty$ *we have*

$$\sum_{N \leq x} a_N = \left(\frac{A(\rho)}{\rho \Gamma(\beta)} + o(1) \right) x^\rho \log(x)^{\beta - 1}.$$

Morally, this is the order of a "Pole" but actually it Can be any Singularity that blows up, like a power of $\frac{1}{s-1}$.

③ Obtain an asymptotic

(this proves the case where $r=1$)

* In our toy example, $\rho = 1/2$ & $\beta = 1/2$.

III. Translating Our Number Theoretic Result to a Geometric Result

Theorem (Linowitz, McReynolds, Pollack, T., 2018)

Fix a number field k, and fix quadratic extensions $L_1, L_2, \ldots, L_r$ of k. Let L be the compositum of the L_i, and suppose that $[L : k] = 2^r$. The number of quaternion algebras over k with discriminant having norm less than x and which admit embeddings of all of the L_i is

$$\sim \delta \cdot x^{1/2} / (\log x)^{1-\frac{1}{2^r}},$$

as $x \to \infty$. Here δ is a positive constant depending only on the L_i and k.

Let $\pi(V, S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 2–orbifolds **derived from quaternion algebras**, each of which has volume less than V and geodesic length spectrum containing S.

finite set

Theorem (Linowitz, McReynolds, Pollack, T., 2018)

If $\pi(V, S) \to \infty$ as $V \to \infty$, then there are integers $1 \leq r, s \leq |S|$ and constants $c_1, c_2 > 0$ such that

$\{\ell_1, \ldots, \ell_r\}$

$$\frac{c_1 V}{\log(V)^{1-\frac{1}{2^r}}} \leq \pi(V, S) \leq \frac{c_2 V}{\log(V)^{1-\frac{1}{2^s}}}$$

for all sufficiently large V.

<u>Pf</u> <u>Sketch</u> → The 3-manifold pf is the same but w/ $PSL_2(\mathbb{C})$ instead

Let M be an arithmetic hyperbolic 2-manifold arising from (K, B) w/ fundamental group $\Gamma < PSL_2(\mathbb{R})$.

There is a bijection:

$$\{c_\gamma : S^1 \to M\} \longleftrightarrow \{[\gamma]_\Gamma : \gamma \in \Gamma\}$$

↑ closed geodesics on M

↑ Γ Conjugacy classes of hyperbolic elts $\gamma \in \Gamma$

Geodesic lengths $\ell(c_\gamma)$ are given by

$$\boxed{\cosh \frac{\ell(c_\gamma)}{2} = \pm \frac{\mathrm{Tr}(\gamma)}{2}}$$

Let $\lambda_\gamma :=$ unique eigenvalue of γ w/ $|\lambda_\gamma| > 1$.

Unique b/c $\det = \pm 1$, $|\mathrm{tr}\,\gamma| > 2$
So one eigenvalue is >1 & one is <1 (in abs. value)

* Each closed geodesic C_γ determines a maximal subfield k_γ of the quaternion algebra B.

$$k_\gamma = k(\lambda_\gamma)$$

(That's how $\ell_1, \ldots, \ell_r$ geodesic lengths in the geometric result correspond to $L_1, \ldots, L_r$ in the number theoretic result)

Let $\pi(V, S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 2–orbifolds **derived from quaternion algebras**, each of which has volume less than V and geodesic length spectrum containing S.

Theorem (Linowitz, McReynolds, Pollack, T., 2018)

If $\pi(V, S) \to \infty$ as $V \to \infty$, then there are integers $1 \leq r, s \leq |S|$ and constants $c_1, c_2 > 0$ such that

$$\frac{c_1 V}{\log(V)^{1-\frac{1}{2^r}}} \leq \pi(V, S) \leq \frac{c_2 V}{\log(V)^{1-\frac{1}{2^s}}}$$

for all sufficiently large V.

Conclusions:

① There are lots of pairwise non-commensurable 3-manifolds w/ a great deal of overlap in their geodesic lengths.

② The counting function looks a bit like the count of prime numbers...