Random walks on weakly hyperbolic groups

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University of Toronto

Random and Arithmetic Structures in Topology
MSRI - Fall 2020

Random walks on weakly hyperbolic groups - Summary

- **Lecture 1** (Aug 31, 10.30): Introduction to random walks on groups

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Random walks, WPD actions, and the Cremona group

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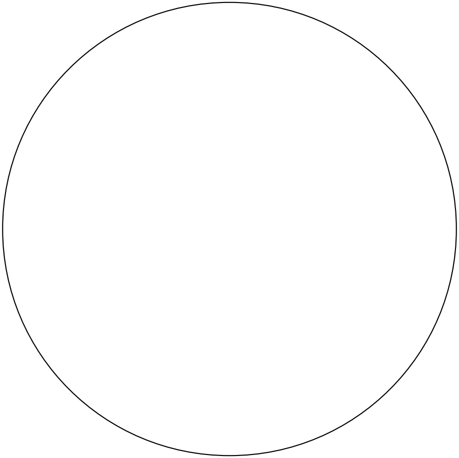

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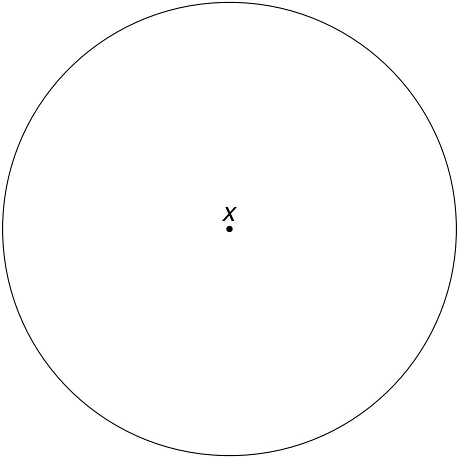

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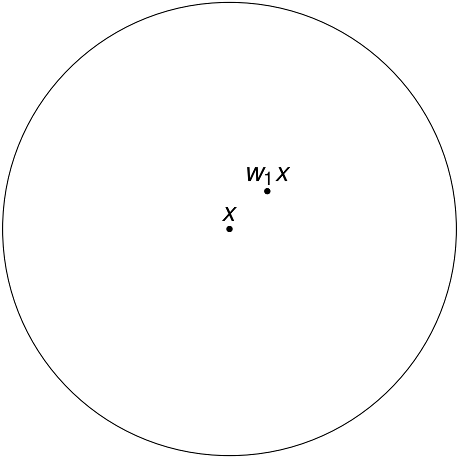

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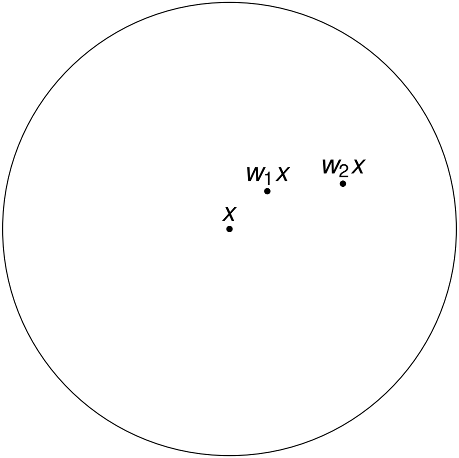

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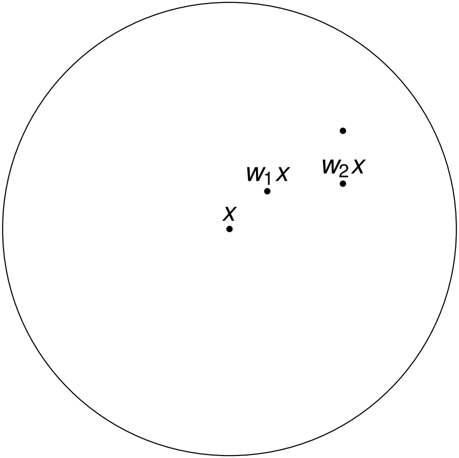

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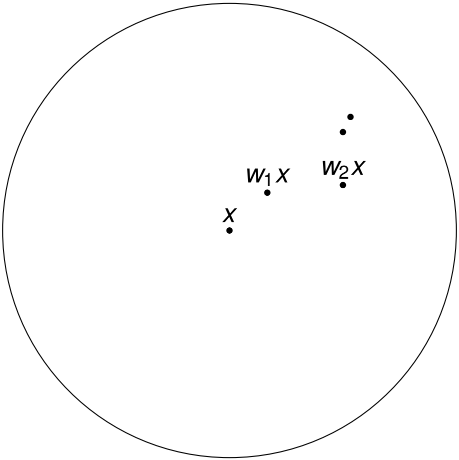

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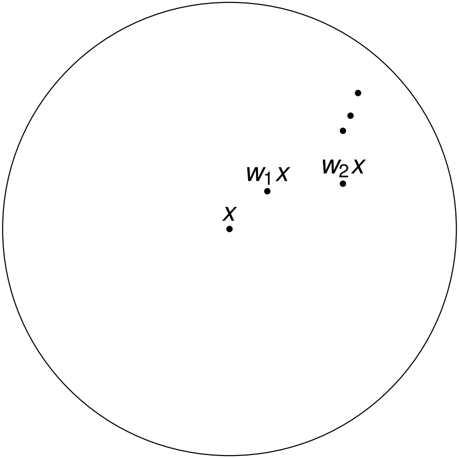

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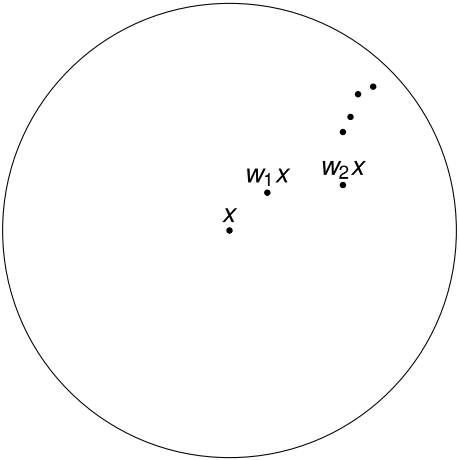

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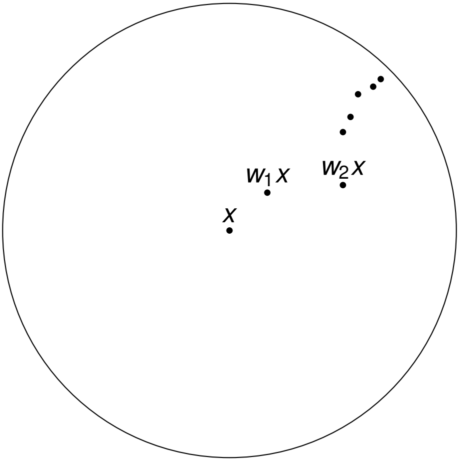

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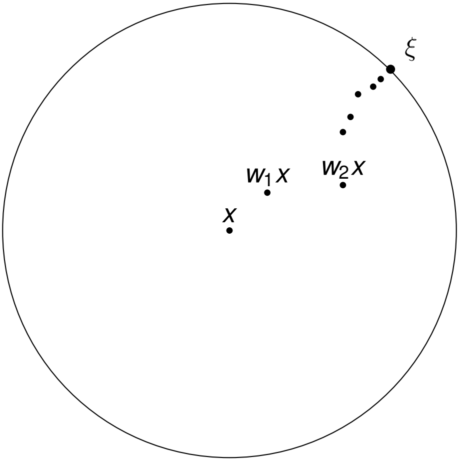

Hyperbolic spaces

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If X is δ-hyperbolic $\implies (x \cdot y)_{x_0} = d(x_0, [x, y]) + O(\delta)$

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Let X be a δ-hyperbolic, non-proper, metric space.
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where the inf is among all finite chains $\xi = \xi_0, \xi_1, \cdots, \xi_{n-1}, \eta = \xi_n$. □

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Definition

The horofunction compactification of (X, d) is the closure

$$\overline{X}^h := \overline{\rho(X)} \qquad \text{in } \mathrm{Lip}^1_{x_0}(X).$$

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Define the action of G on $\overline{X}^h$ as

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If X is separable, then the horofunction compactification $\overline{X}^h$ is a compact metrizable space.

Proof.

If $h \in \mathrm{Lip}^1_{x_0}(X)$, then

$$|h(x)| \leq |h(x) - h(x_0)| \leq d(x, x_0)$$

hence $\mathrm{Lip}^1_{x_0}(X) \subset \otimes_{x \in X}[-d(x, x_0), d(x, x_0)]$ which is compact by Tychonoff's theorem.
Since X is separable, then $C(X)$ is second countable,
hence $\overline{X}^h$ is second countable.
Thus $\overline{X}^h$ is compact, Hausdorff, and second countable, hence metrizable. □

Definition

Define the action of G on $\overline{X}^h$ as

$$g.h(x) := h(g^{-1}x) - h(g^{-1}x_0) \qquad \text{for all } g \in G, h \in \overline{X}^h.$$

Examples of horofunctions - I

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hence $\partial^h X = \overline{X}^h \setminus X = \{-\infty, +\infty\}$.

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the usual definition of horofunction, and level sets are horoballs.

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(**Note**: the set of infinite horofunctions is NOT closed.)

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where d^+ is the oriented distance along the geodesic, for some choice of orientation of γ.

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Since every horofunction is the pointwise limit of functions ρ_z, the claim follows. □

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E.g., in the "infinite tree" case, if $z_n := (n, n)$ then $\rho_{z_n} \to \rho_{x_0}$ but $\phi(\rho_{z_n}) = z_n \not\to x_0$.

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$P(\overline{X}^h)$ is compact, so it contains a μ-stationary measure.

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As a corollary, for $\mathbb{P}$-a.e. sample path (w_n) there exists a subsequence $(w_{n_k} x_0)$ which converges to a point in the Gromov boundary ∂X.

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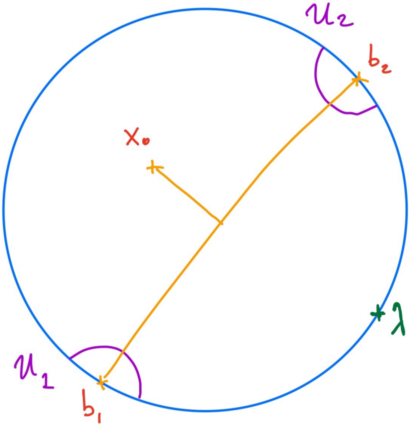

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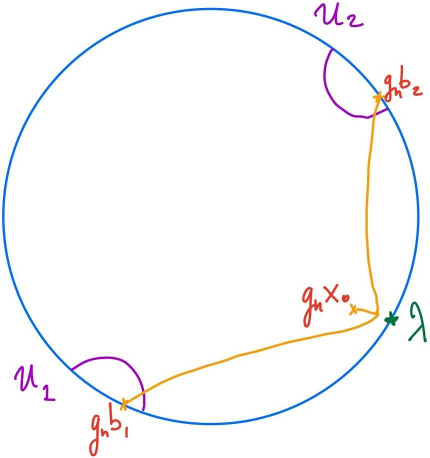

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□

Corollary

The hitting measure is the only μ-stationary measure on ∂X.