Strong contractability of geodesics in PMCG(S)

Yvon Verberne with Kasra Rafi
MSRI - Random and Arithmetic Structures in Topology

Synopsis

Conjecture (Farb): The set of pseudo-Anosov elements is generic with respect to the word metric.

Theorem (Yang): If one pseudo-Anosov element in $\mathrm{MCG}(S)$ has the strong contracting property, Farb's conjecture holds

Conjecture (People we talked to): Every pseudo-Anosov element should have the strong contracting property

Rafi-V.: Found a pseudo-Anosov element which does not have the strong contracting property

Outline

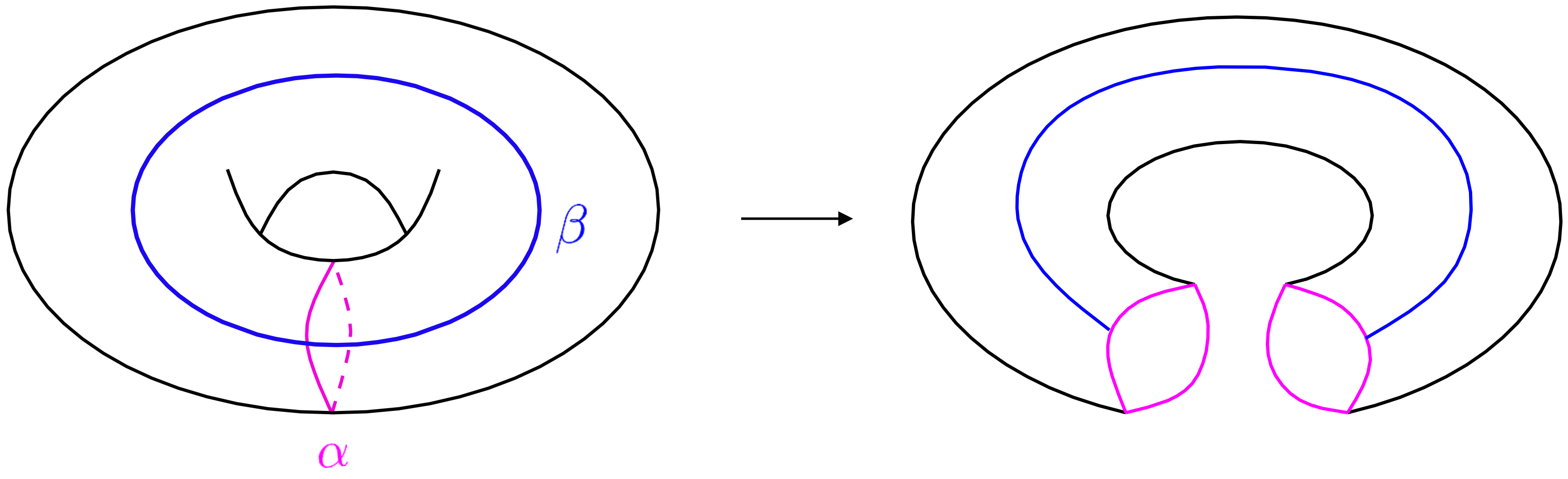
1. Introducing mapping class groups
2. History of the problem
3. Introducing main theorem
4. Proof sketch of main theorem

The mapping class group

Mapping class groups

$$\mathrm{MCG}(S) = \mathrm{Homeo}^+(S)/\mathrm{isotopy}$$

Dehn-Lickorish: finitely generated by Dehn twists, D_α



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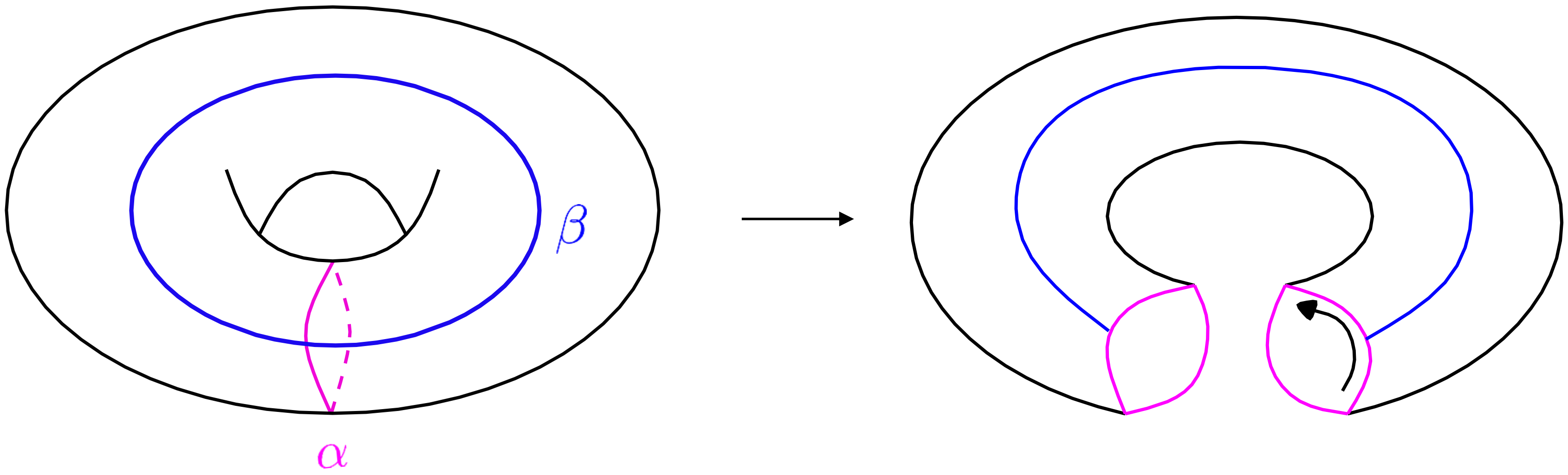

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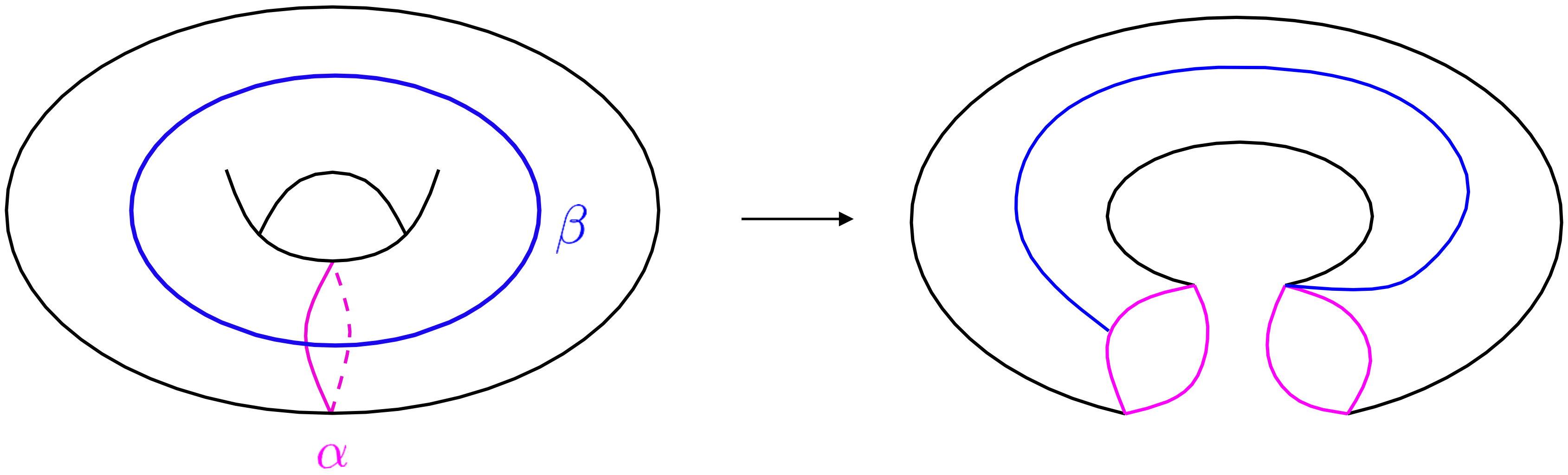

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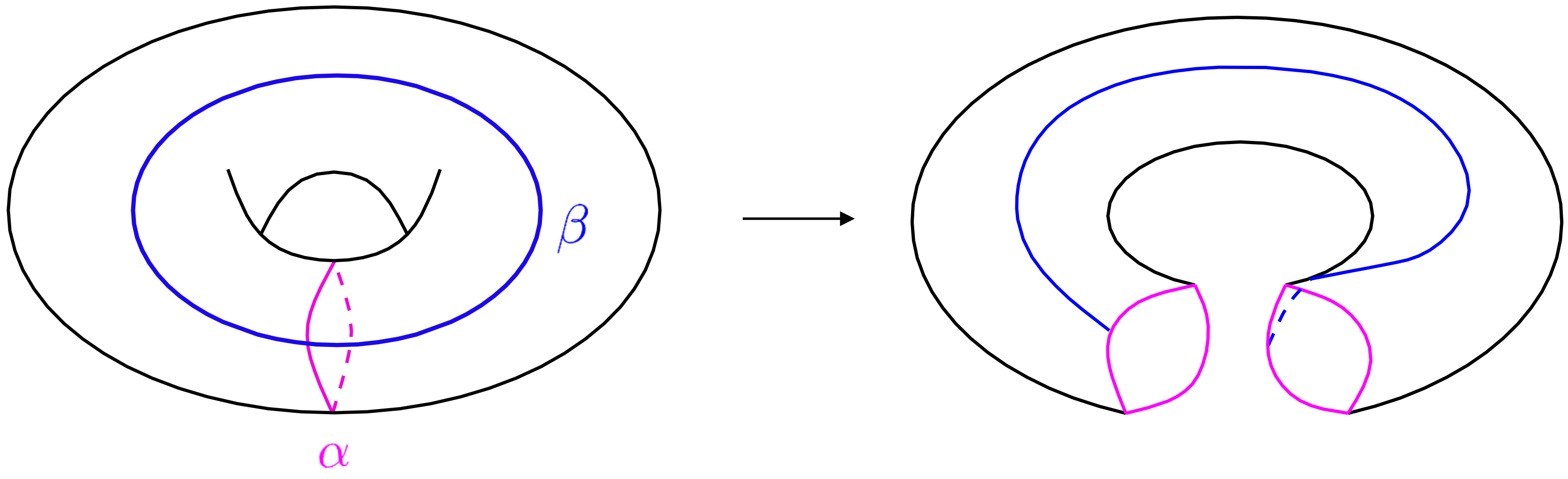

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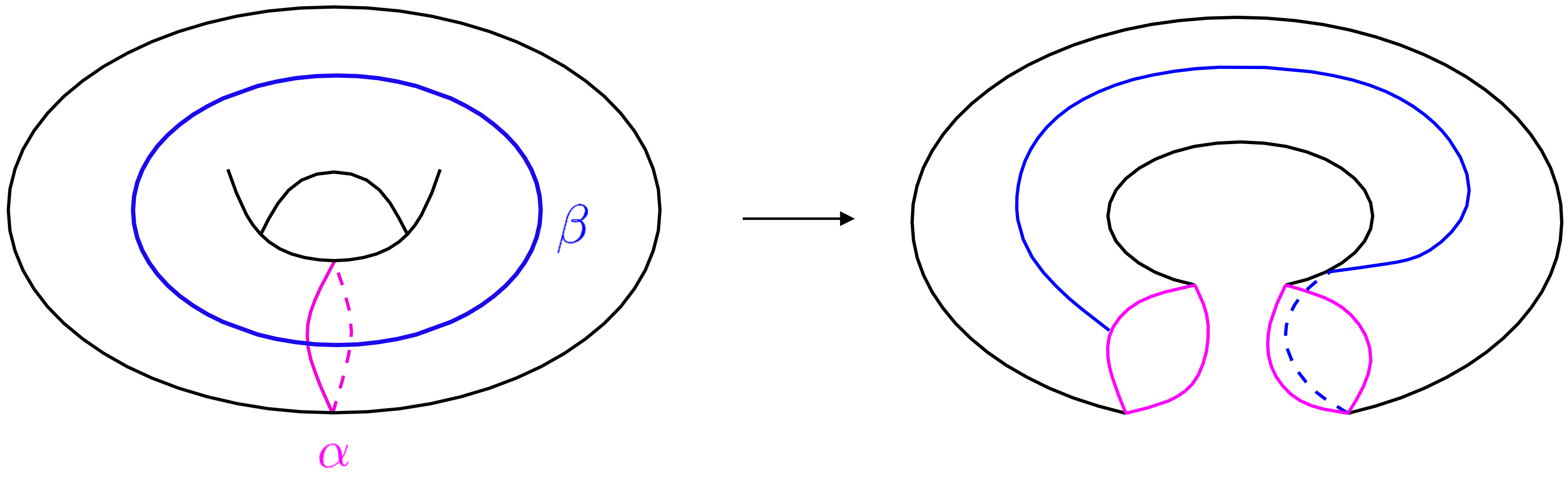

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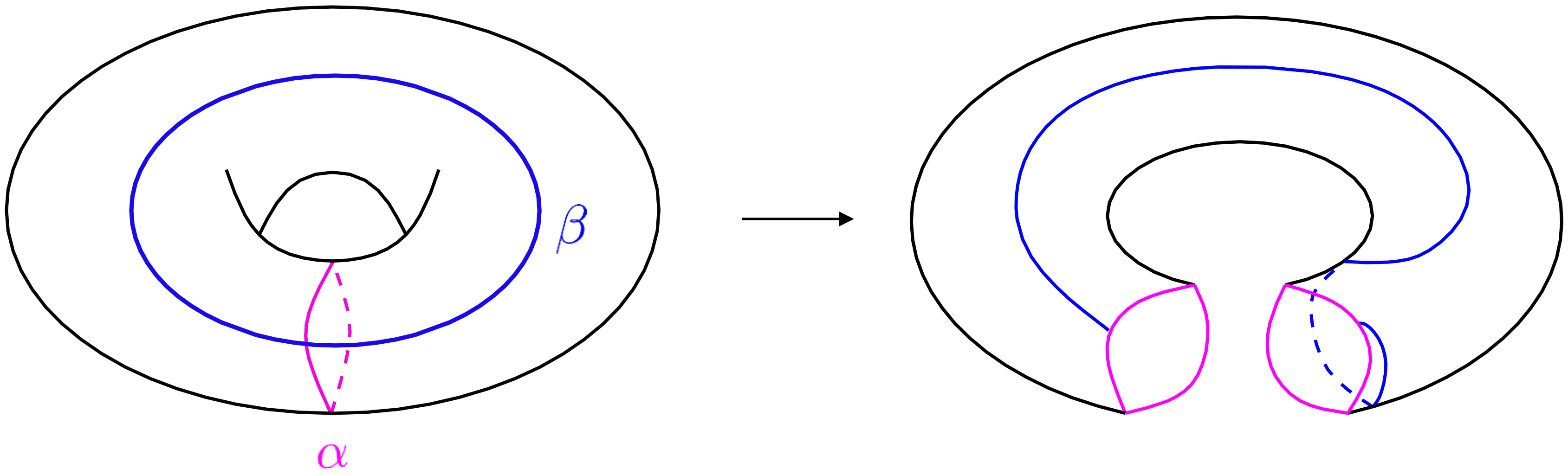

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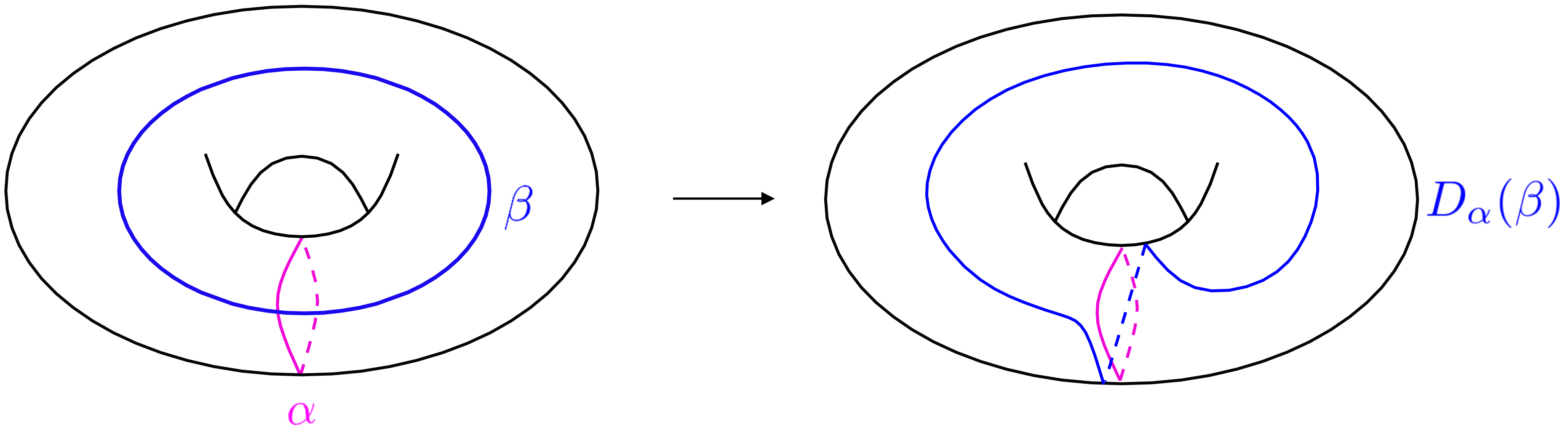


Positive twist: twist in right direction

Mapping class groups

Dehn-Lickorish: finitely generated by Dehn twists, D_α

Cayley graph:

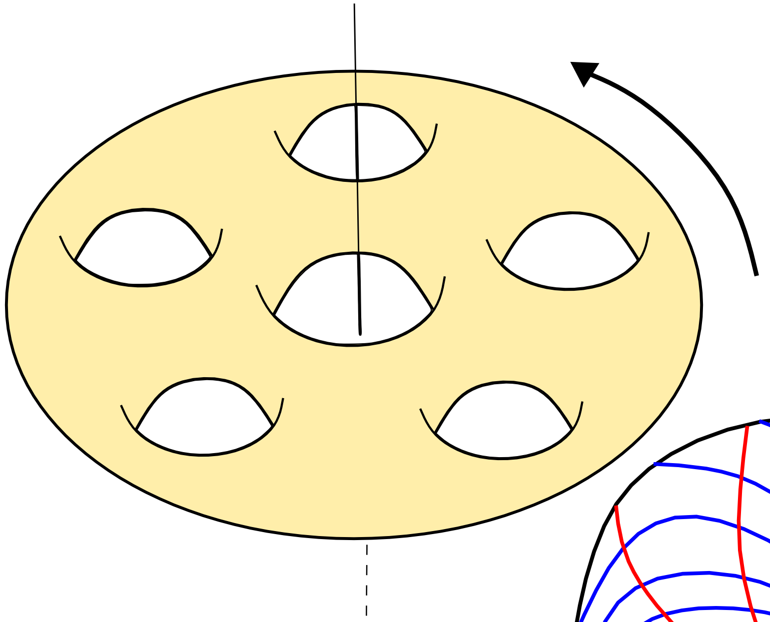
Choose a generating set, $\mathcal{S} = \{s_1, \ldots s_n\}$

Vertices = elements in mapping class group

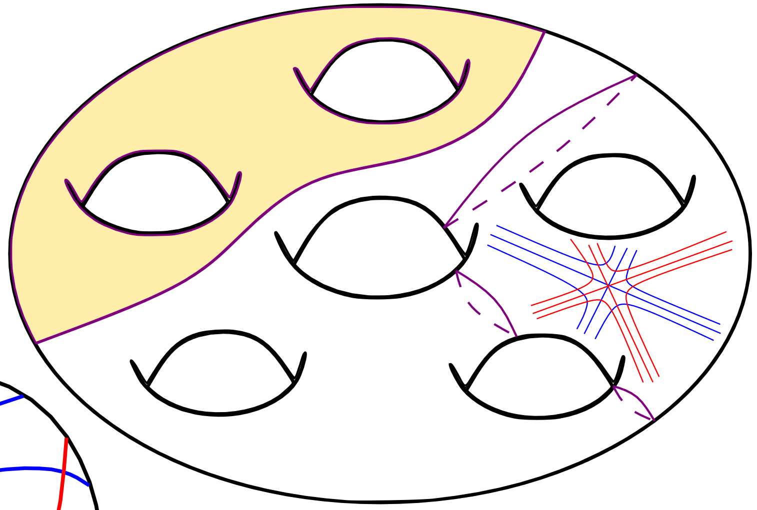
Edges = when two elements differ by one element of $\mathcal{S}$

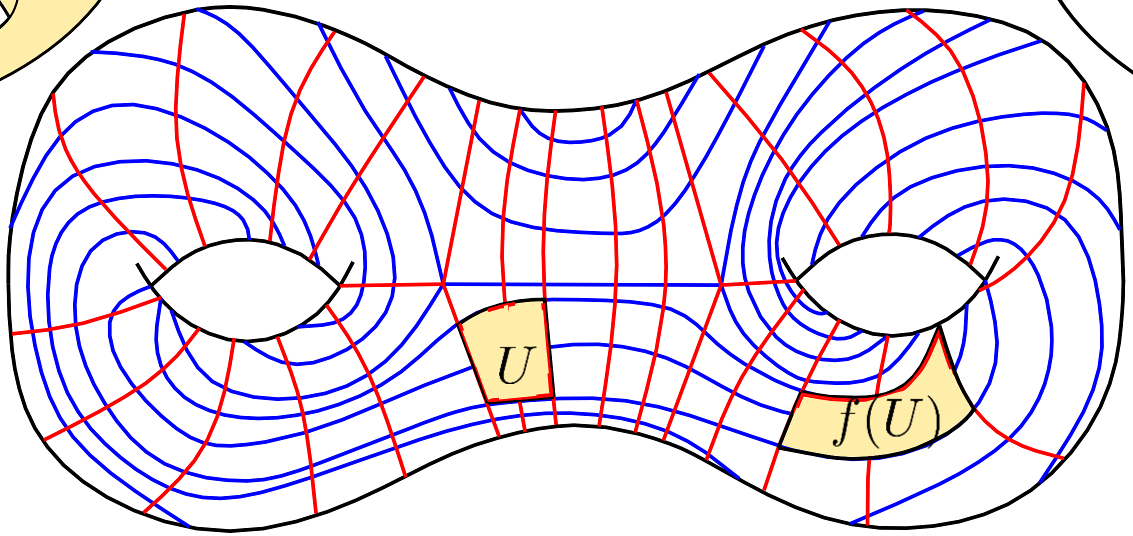
Cayley graph $\rightsquigarrow$ metric on $\mathrm{MCG}(S)$

Nielsen-Thurston Classification

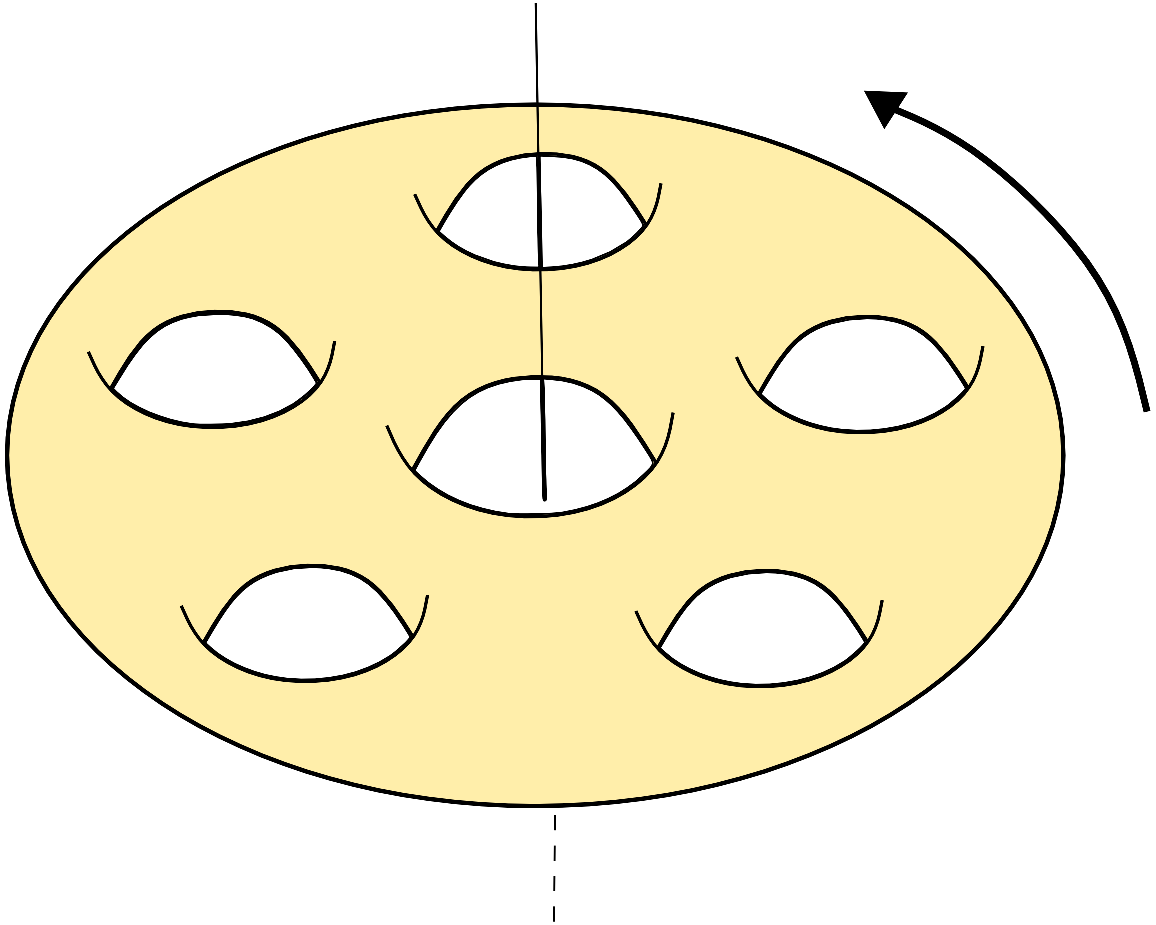


$f \in \mathrm{MCG}(S)$



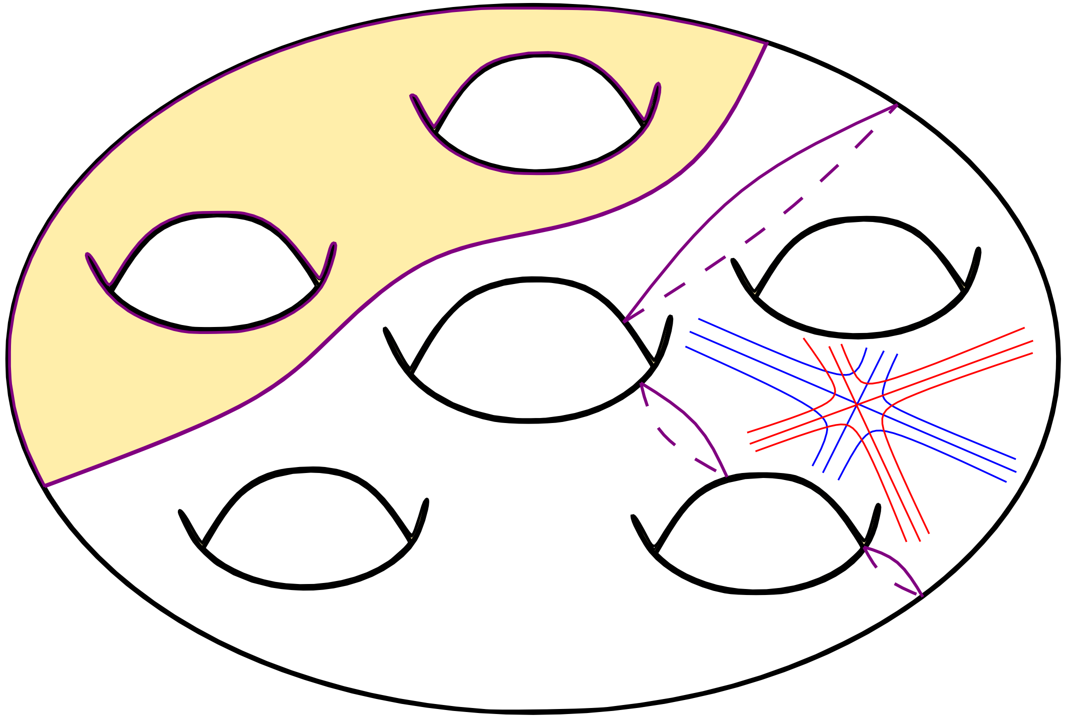


Periodic



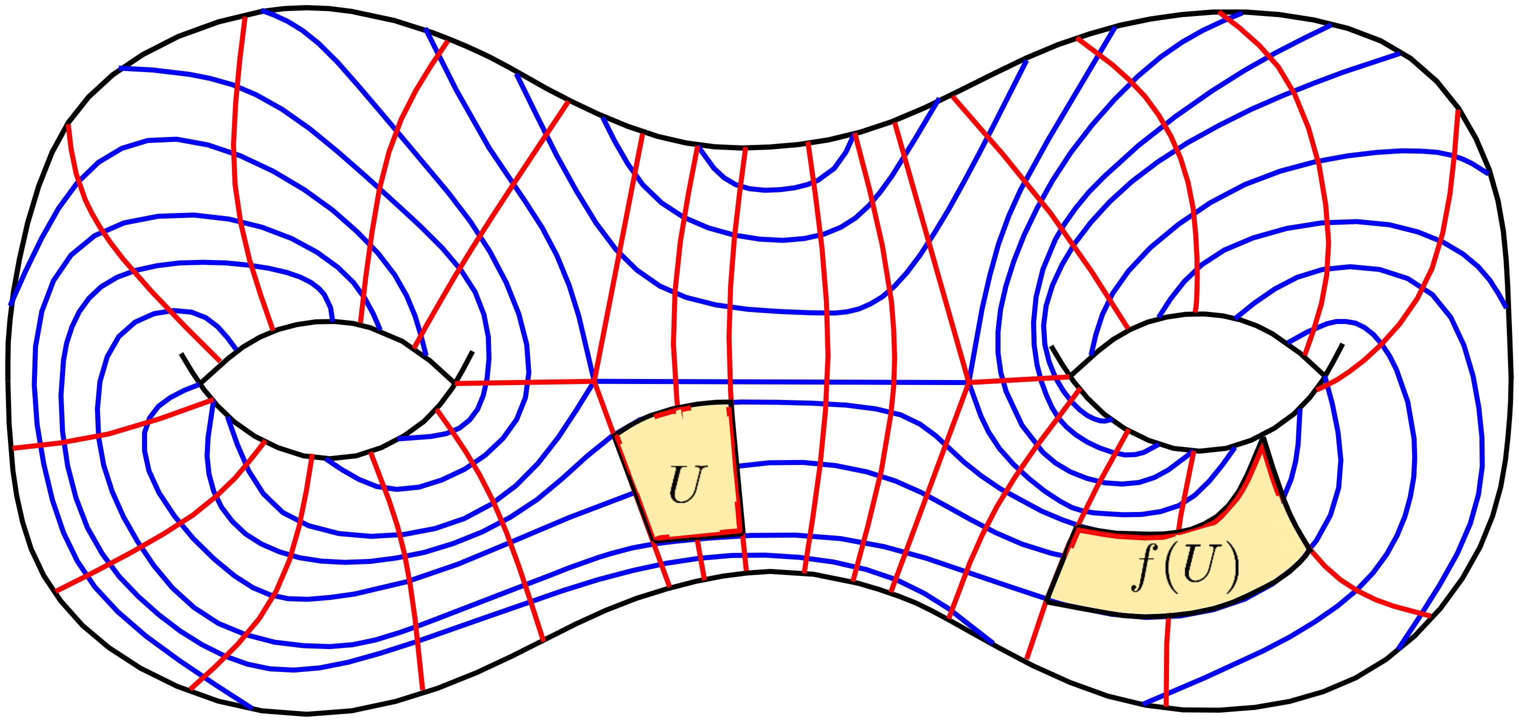
f has finite order

Reducible



Nonempty set $\{c_1, \ldots c_n\} \in S$ so that $i(c_j, c_k) = 0$ for $j \neq k$ and so that $\{f(c_j)\} = \{c_j\}$.

Pseudo-Anosov



f maps no curve back to itself

Theorem (Thurston):

$\exists$ a number $\lambda > 1$ and a pair of foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ such that $f(\mathscr{F}^u) = \lambda \mathscr{F}^u$ and $f(\mathscr{F}^s) = \lambda^{-1} \mathscr{F}^s$.

Aside: Applications to number theory

The Mahler Measure of a monic integral polynomial is the product of the absolute value of roots outside the unit circle.

Problem (Lehmer): The Mahler measure of a polynomial can be arbitrarily close but not equal to 1.

Leininger: Found a pseudo-Anosov whose dilatation (stretch factor) is equal to Lehmer's number.

Silver-Williams: Lehmer's question is equivalent to one about generalized growth rates of Lefschetz numbers of iterated pseudo-Anosov surface homeomorphisms.

Aside: Applications to number theory

A biPerron number is a real algebraic integer $\lambda > 1$ such that all Galois conjugates z of λ except λ^{-1} are contained in the annulus $\lambda^{-1} < |z| < \lambda$.

Problem (Fried): If λ is a biPerron number of norm ± 1 (i.e. an algebraic unit), some power of λ is the stretch factor of a pseudo-Anosov mapping class.

Kenyon: Proved Fried's conjecture for degree 3.

Constructed pseudo-Anosov mapping classes on the 3-torus.

Aside: Applications to number theory

A real algebraic unit $\lambda > 1$ is a Salem number if λ^{-1} is a Galois conjugate, and all other conjugates lie on the unit circle.

Problem: Can we get every Salem number as a pseudo-Anosov stretch factor?

Pankau: For any Salem number λ, there exists an integer k so that λ^k is a pseudo-Anosov stretch factor.

Aside: Applications to 3-manifolds

Nielsen-Thurston: Mapping classes are either periodic, reducible, or pseudo-Anosov.

The mapping torus for $\phi \in \mathrm{MCG}(S_g)$ is $M_\phi = \frac{S_g \times [0,1]}{(s,0)\sim(\phi(x),1)}$.

Theorem (Thurston): Let M_ϕ be the mapping torus for $\phi \in \mathrm{MCG}(S_g)$.

- ϕ periodic iff M_ϕ admits a metric locally isometric to $\mathbb{H}^2 \times \mathbb{R}$.
- ϕ reducible iff M_ϕ contains an incompressible torus.
- ϕ pseudo-Anosov iff M_ϕ admits a hyperbolic metric.

History of problem

What is a generic element in $\mathrm{MCG}(S)$?

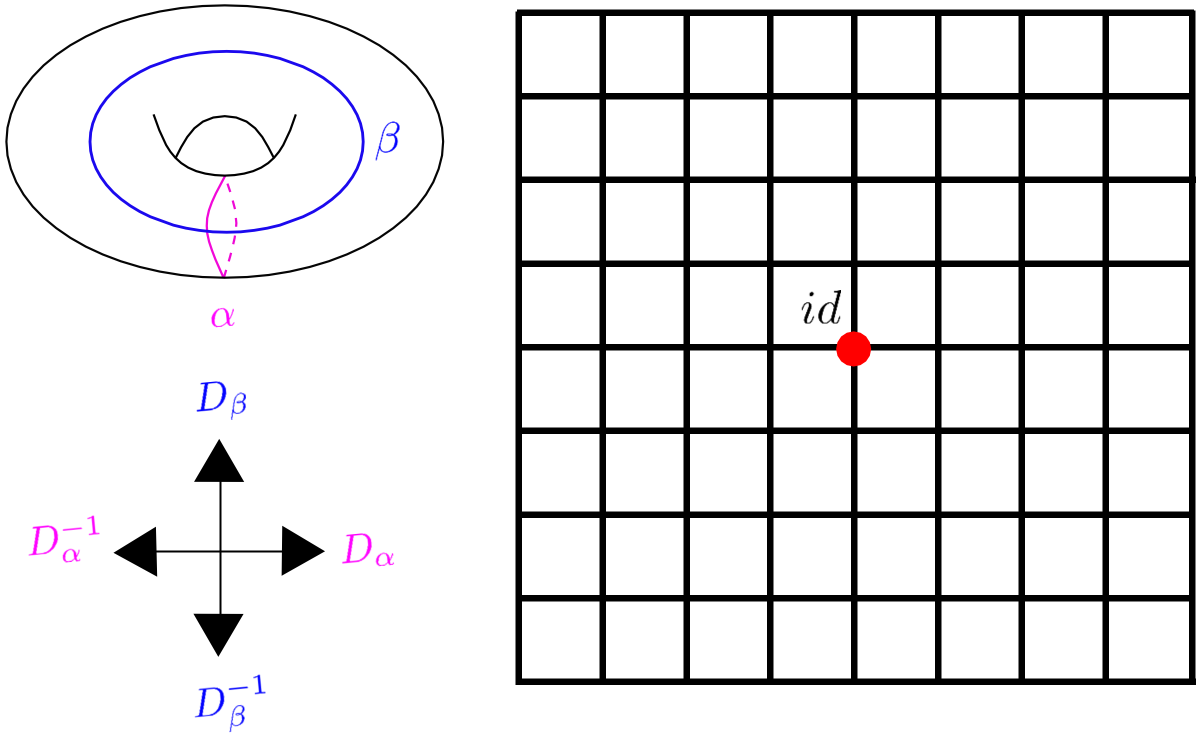
Two Notions of Genericity

1. With respect to random walks

2. With respect to the word metric

Genericity with respect to random walks

Consider a random walk on $\mathrm{MCG}(S)$.



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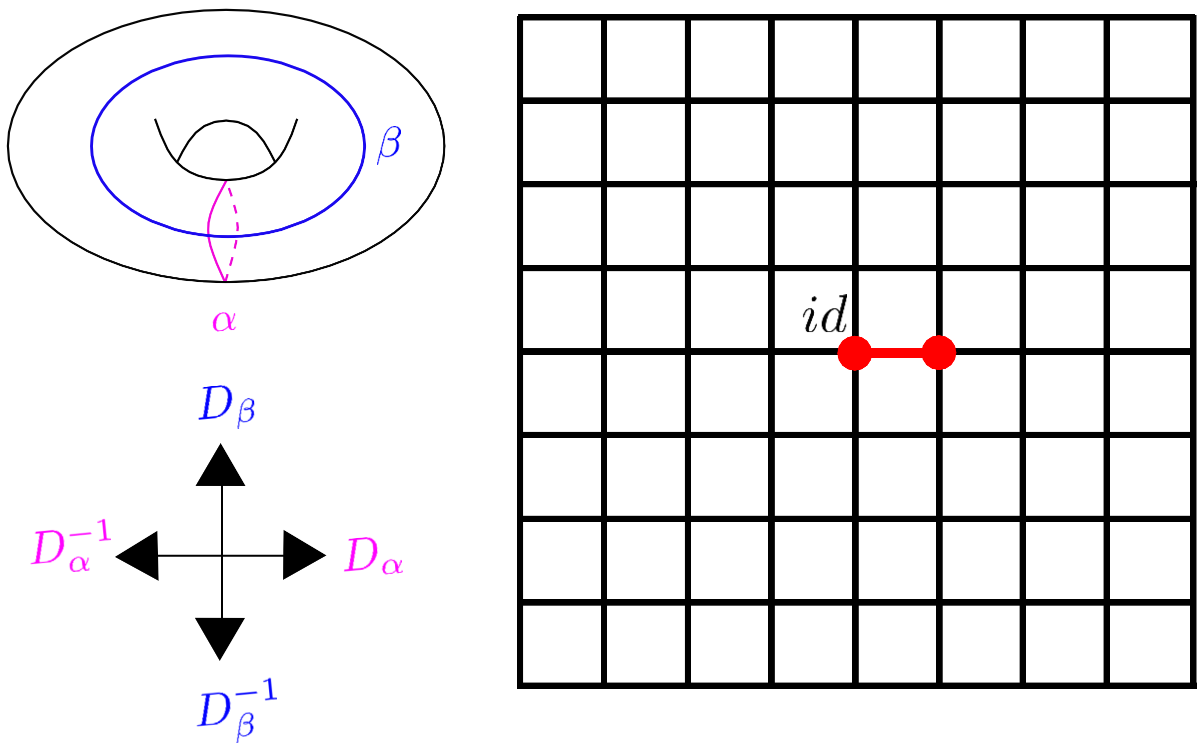

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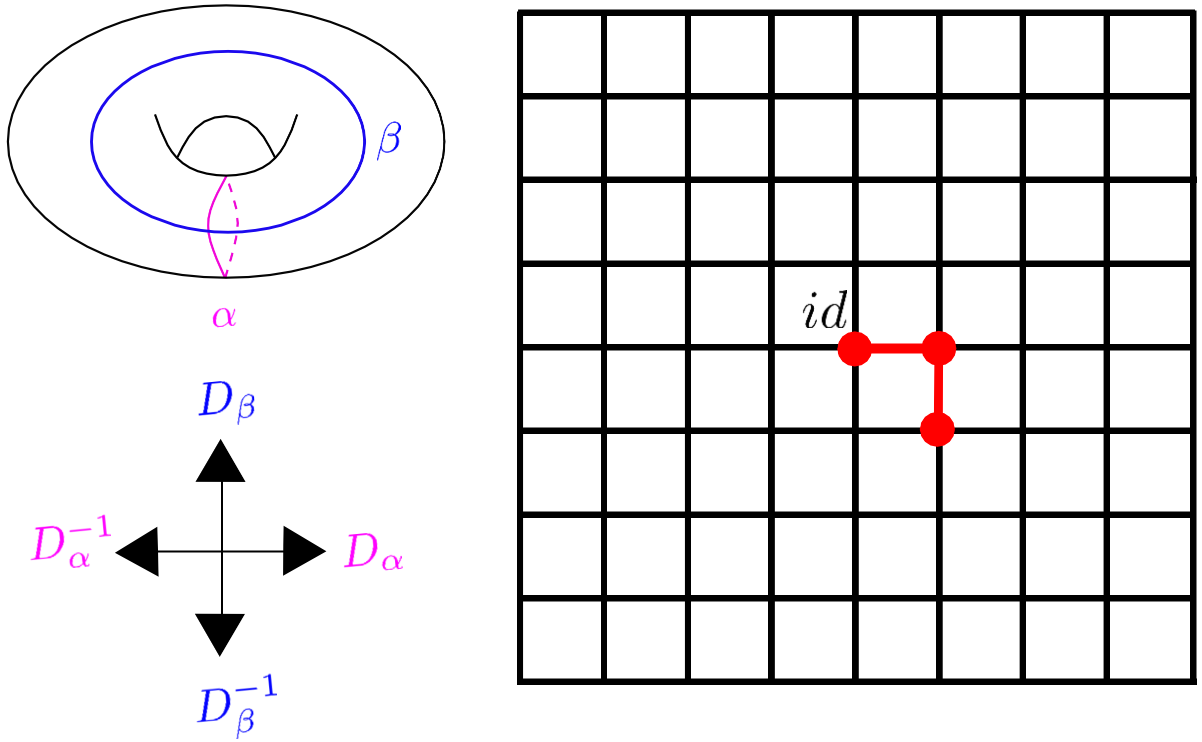

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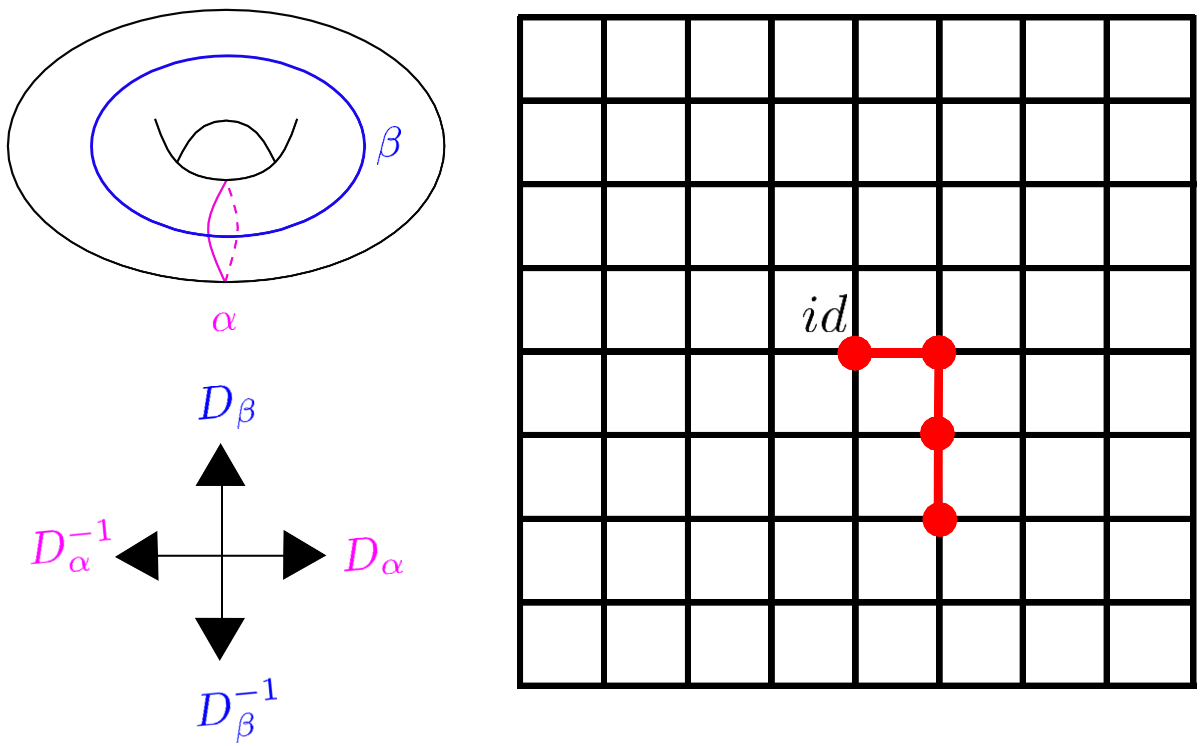

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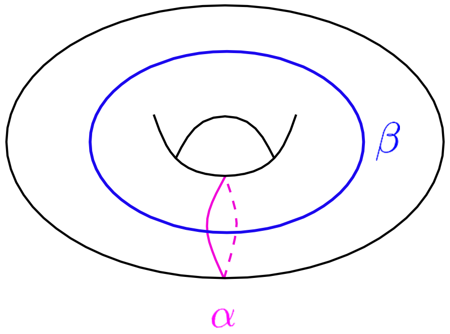

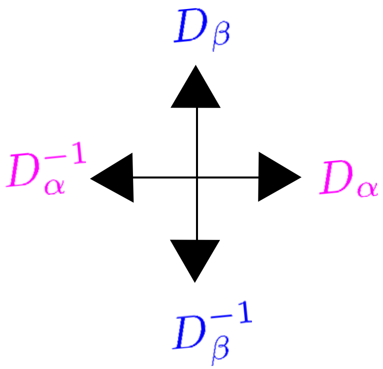

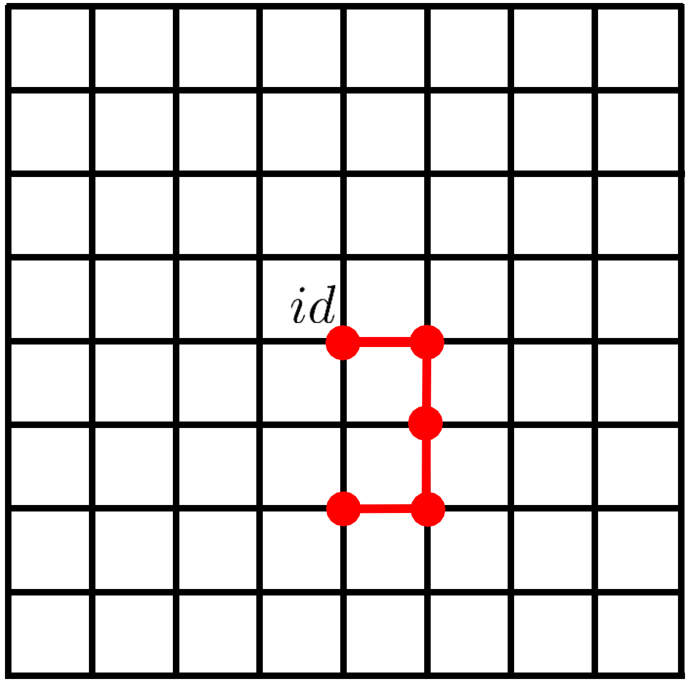

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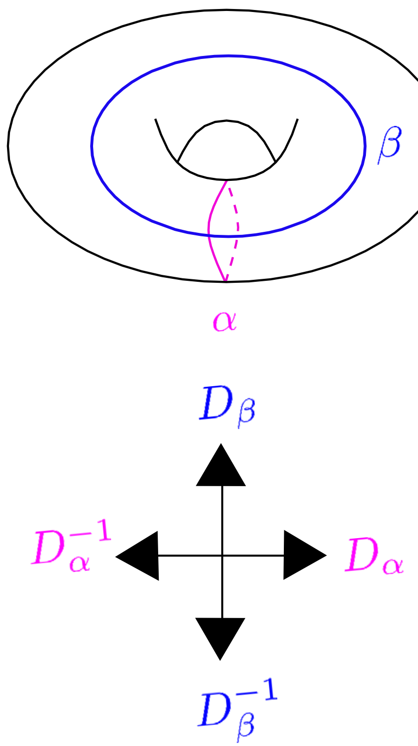

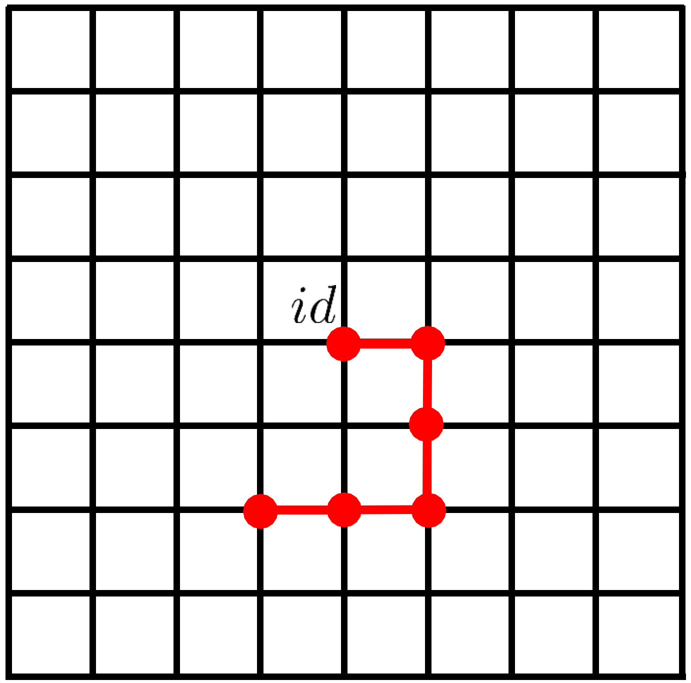

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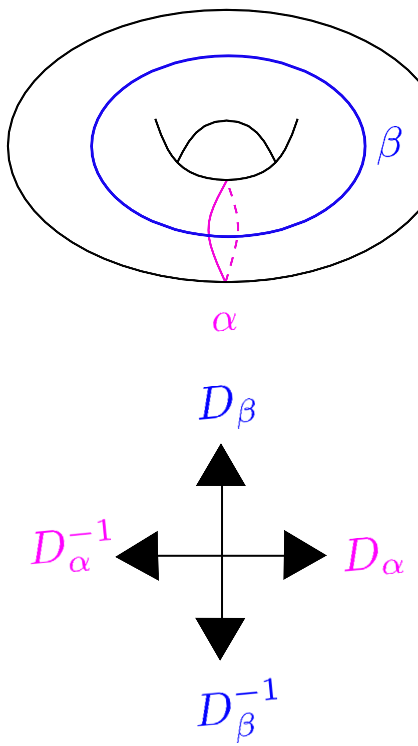

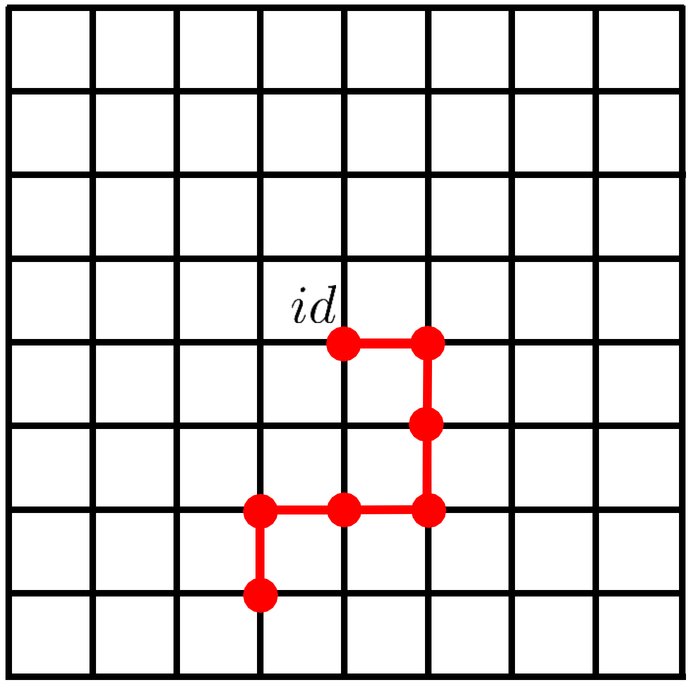

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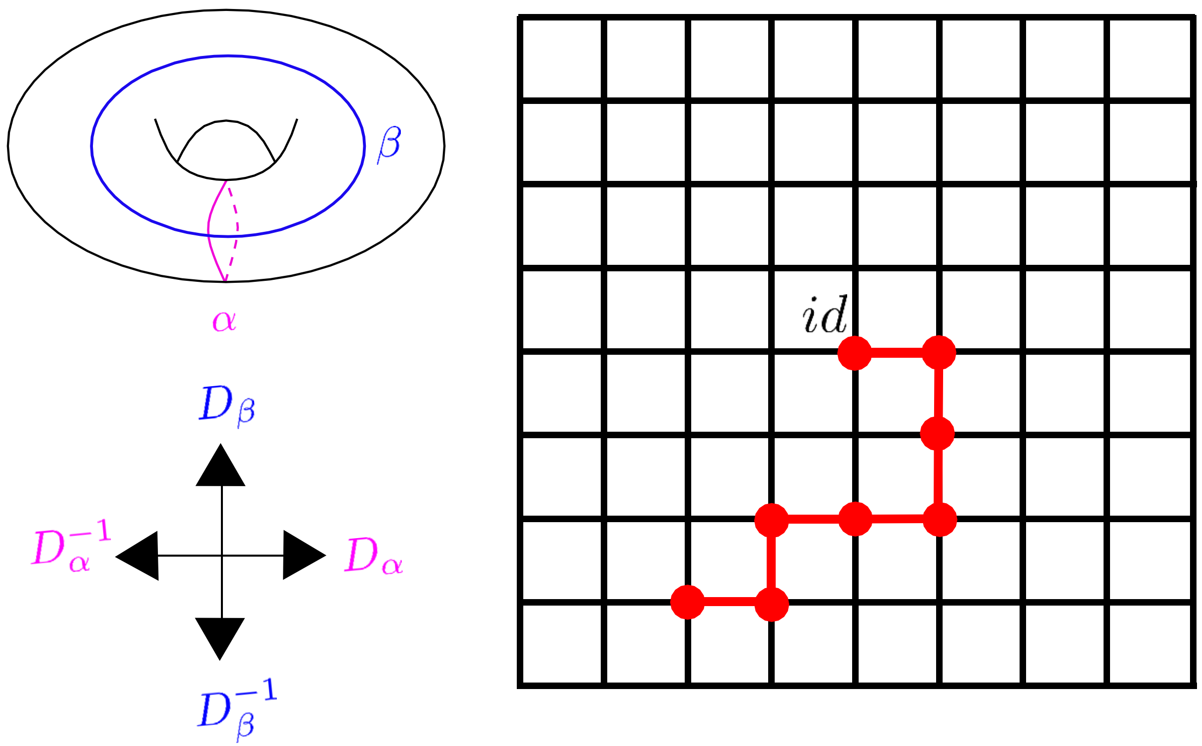

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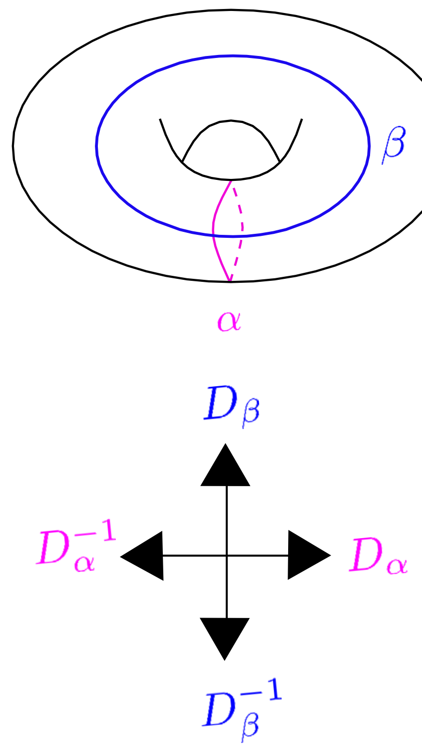

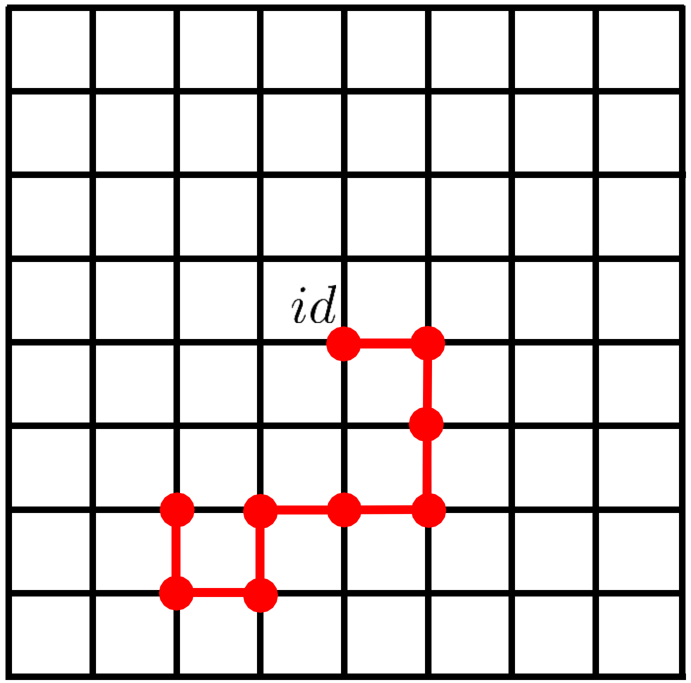

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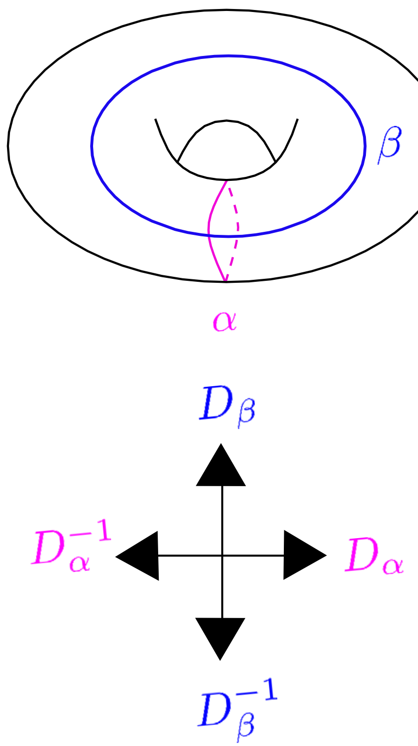
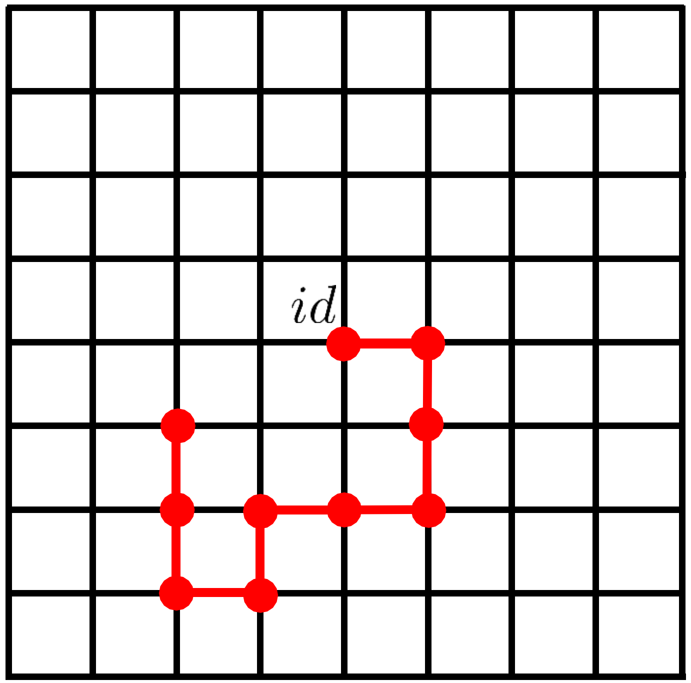




Question: Is the n-th step of a random walk on the mapping class group of a surface almost surely a pseudo-Anosov element as n goes to infinity?

Answer (Maher): Yes!

Genericity with respect to random walks

Consider a random walk on $\mathrm{MCG}(S)$ where we consider a symmetric generating set

Question: Does the probabability of a random walk on $\mathrm{MCG}(S)$ not being pseudo-Anosov decay exponentially with respect to the length of the word?

Answer (Riven): Yes!

Answer (Malestein-Suoto): This is also true for the Torelli subgroup!

The Torelli subgroup of the mapping class group are the elements of $\mathrm{MCG}(S)$ which act trivially on the integer homology of S.

Genericity with respect to random walks

Question (Margalit): Is the generic element of $\mathrm{Mod}(S_g)$ a normal generator, or not?

Consider a random walk on a group which acts on a hyperbolic space.

Theorem (Maher-Tiozzo): The probability that the normal closure $<< w_n >>$ of w_n in G is free satisfies

$$\mathbb{P}(<< w_n >> \text{ is free}) \to \frac{1}{k}$$

as $n \to \infty$. As a Corollary, this probability can tend to 1 in special cases.

Answer (Maher-Tiozzo): A random normal subgroup of the mapping class group is free.

Genericity with respect to the word metric

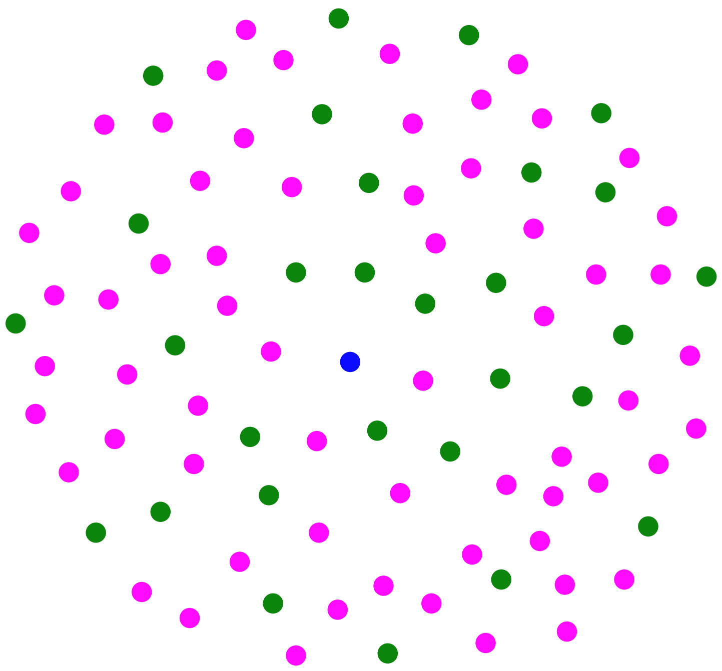
Conjecture (Farb):

The set of pseudo-Anosov elements is generic with respect to the word metric.

Generating set $\rightsquigarrow$ Cayley graph

Denote $B_r := \{g \in \mathrm{MCG}(S) : \|g\| \leq r\}$.

$X \subseteq \mathrm{MCG}(S)$ is generic if $\lim_{r\to\infty} \dfrac{\#B_r \cap X}{\#B_r} = 1.$



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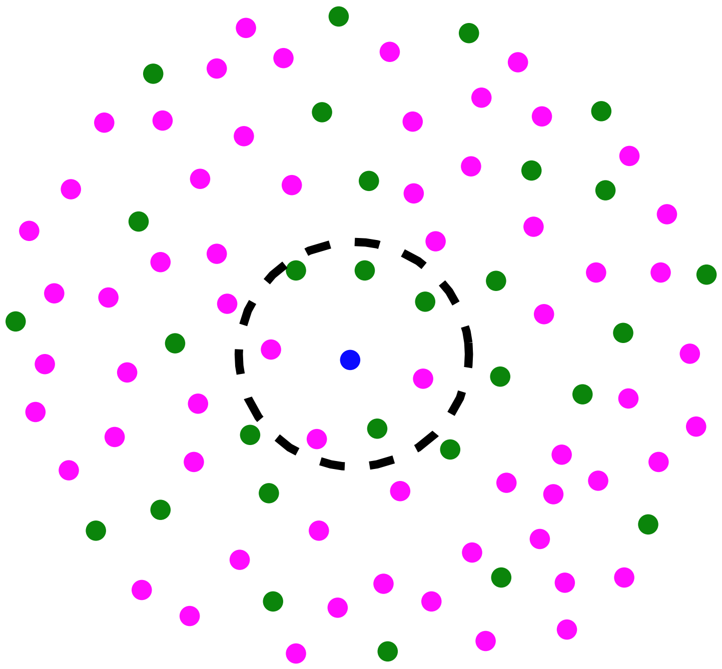

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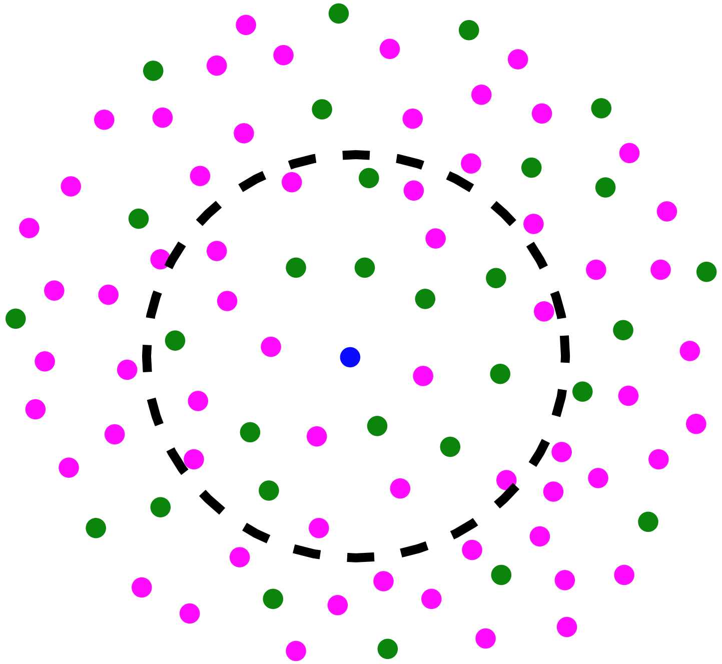

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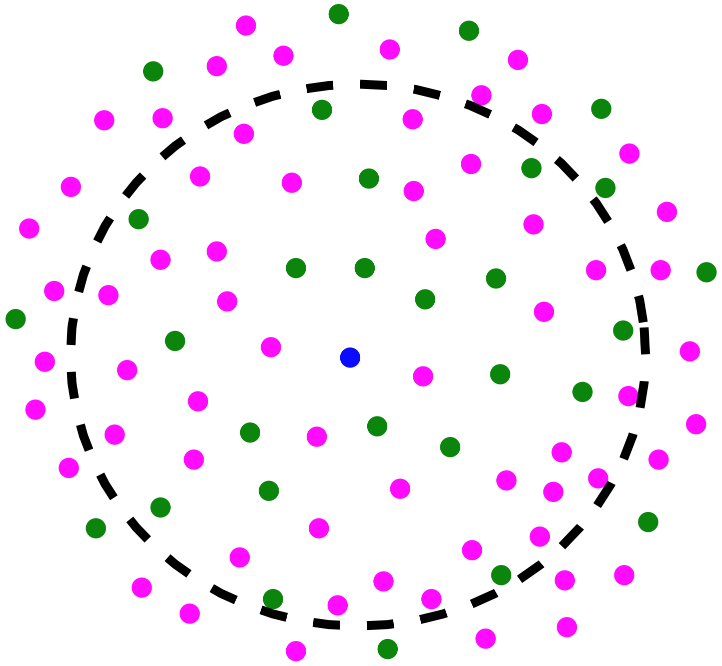

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Genericity with respect to the word metric

Prior Results

Dani: Determined a formula for $\lim_{r\to\infty} \frac{\#B_r \cap X}{\#B_r}$ for a virtually nilpotent group G and $X \subset G$ is the subgroup of finite-order elements.

Cumplido-Weist: In the case of the mapping class group where $X \subset \mathrm{MCG}(S)$ the subset of pseudo-Anosov elements, determined $\lim_{r\to\infty} \frac{\#B_r \cap X}{\#B_r}$ stays bounded away from 0.

Genericity with respect to the word metric

A new direction

Theorem (Yang): If one pseudo-Anosov element in $\mathrm{MCG}(S)$ has the strong contracting property, Farb's conjecture holds.

Conjecture (People we talked to): Every pseudo-Anosov element should have the strong contracting property.

Main Result (Rafi-V.)

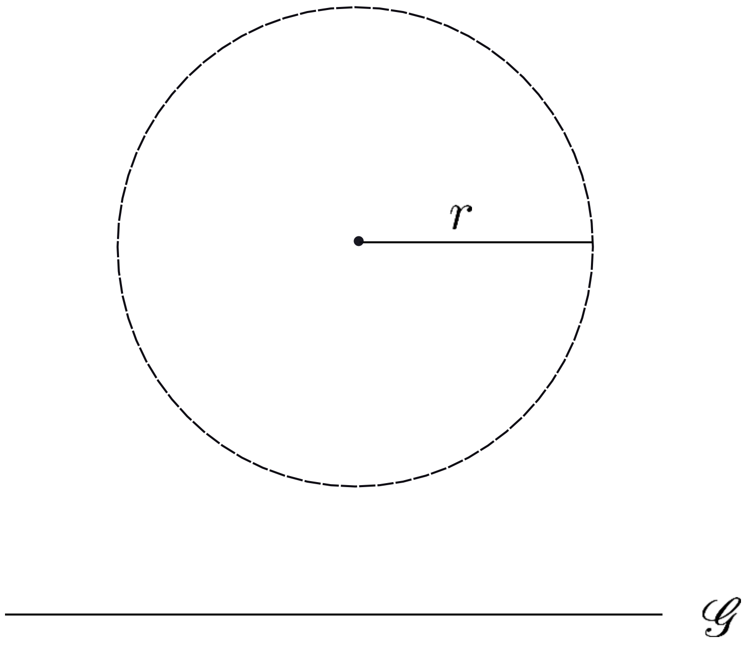
Let $S = S_{0,5}$.

Constructed pseudo-Anosov which is not strongly contracting in $\mathrm{PMCG}(S)$.

$\mathrm{PMCG}(S) \subseteq \mathrm{MCG}(S)$ fixes punctures individually.

Contracting property

A geodesic, $\mathscr{G}$, is contracting if there exists $\rho < 1$ and $c < \infty$ such that for any ball such that $\mathscr{G} \cap B_r(x) = \emptyset$, $\operatorname{diam}(\operatorname{Proj}(B_{\rho r}(x))) \leq c$.



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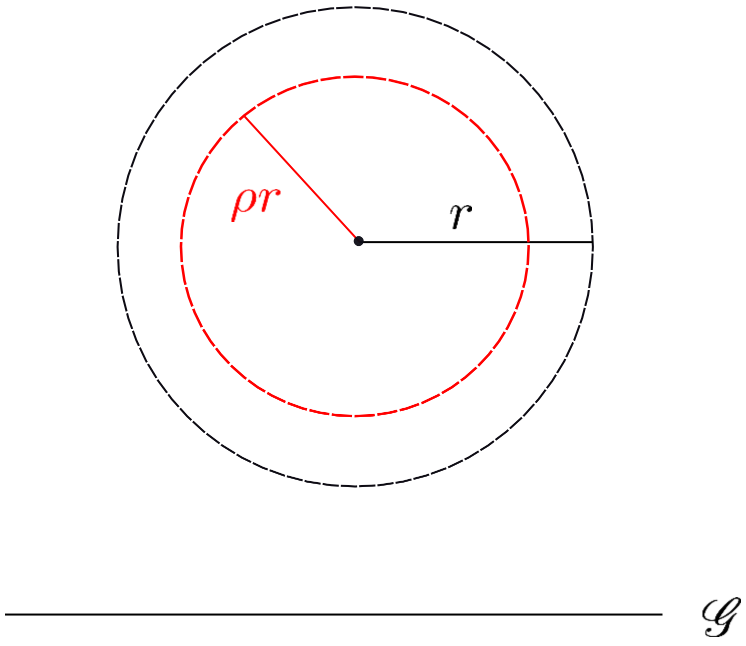

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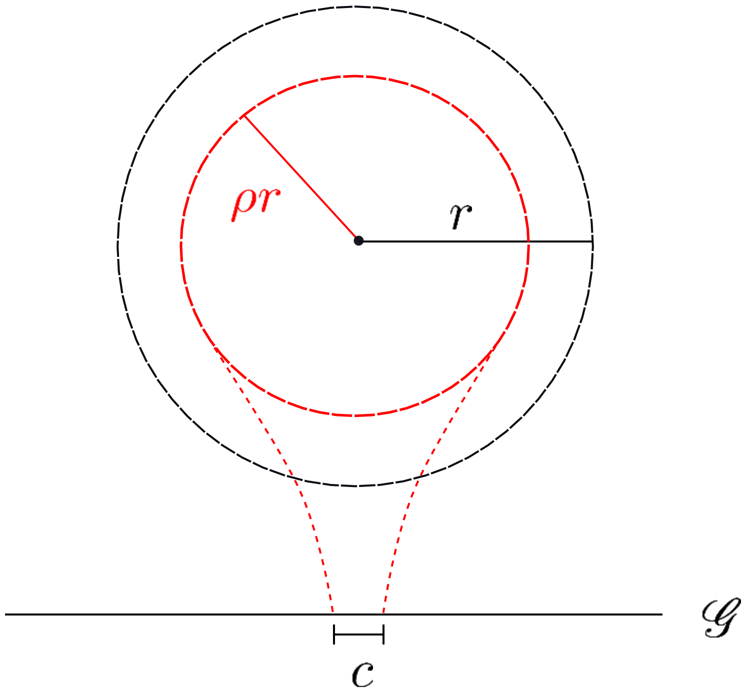

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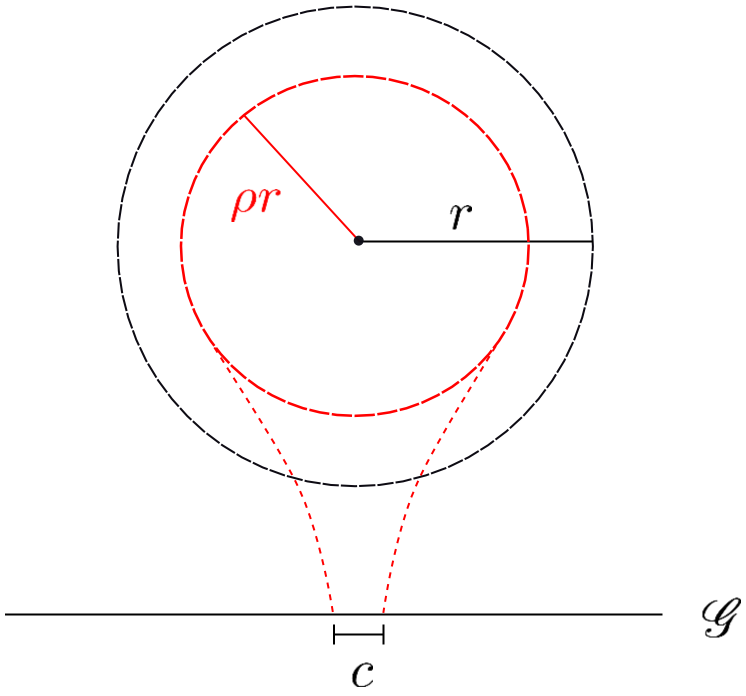
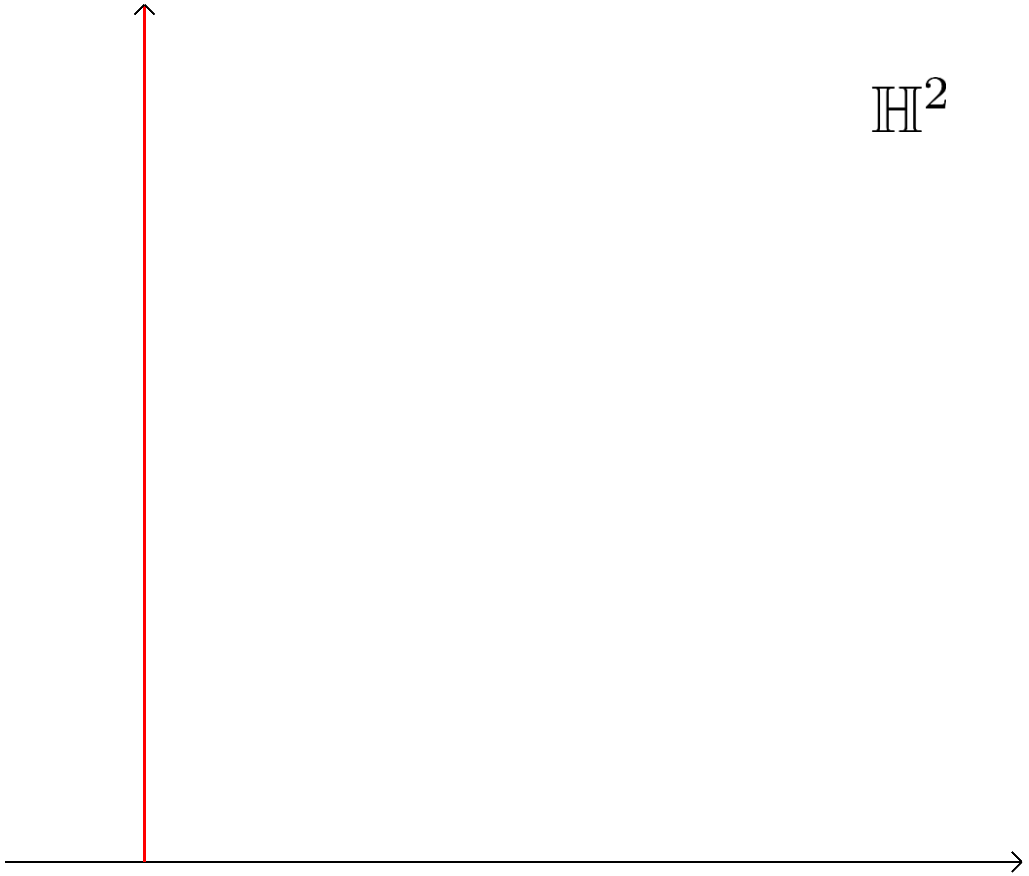
A geodesic, $\mathscr{G}$, is strongly contracting if $\rho = 1$.

Every geodesic of $\mathbb{H}^2$ is strongly contracting.



Contracting property

Every geodesic of $\mathbb{H}^2$ is strongly contracting (Arzhantseva-Cashen-Gruber-Hume).



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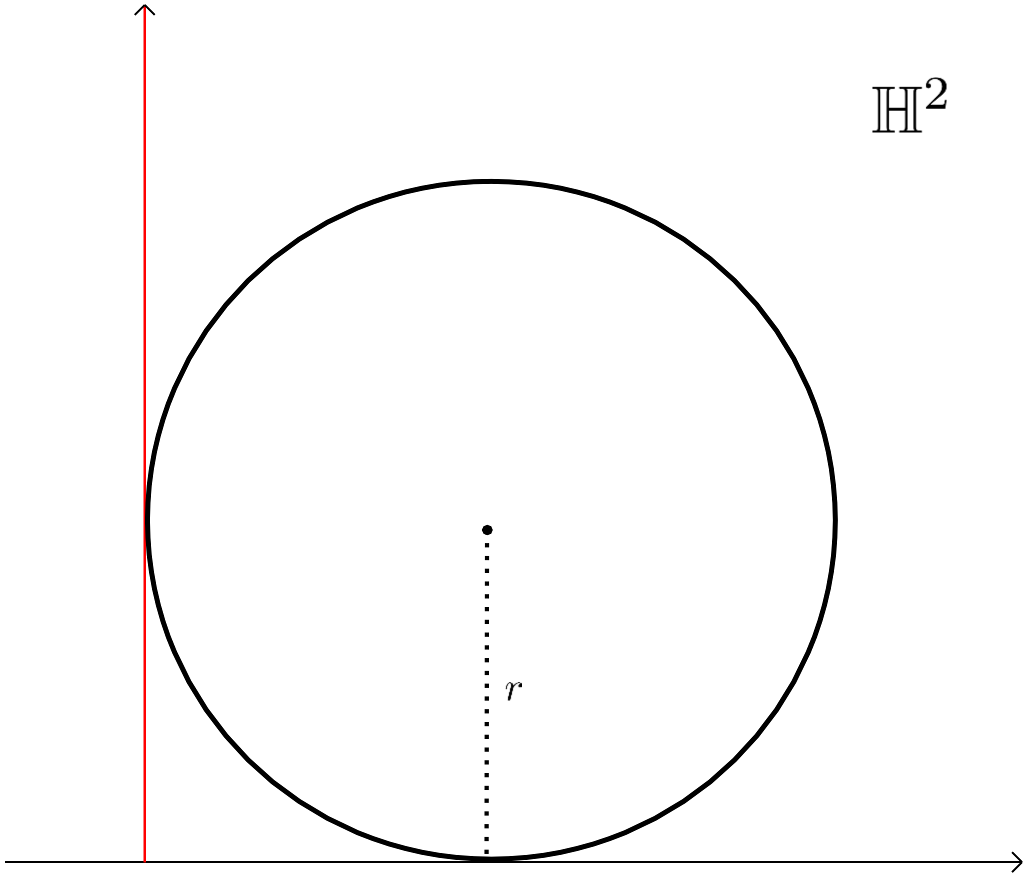

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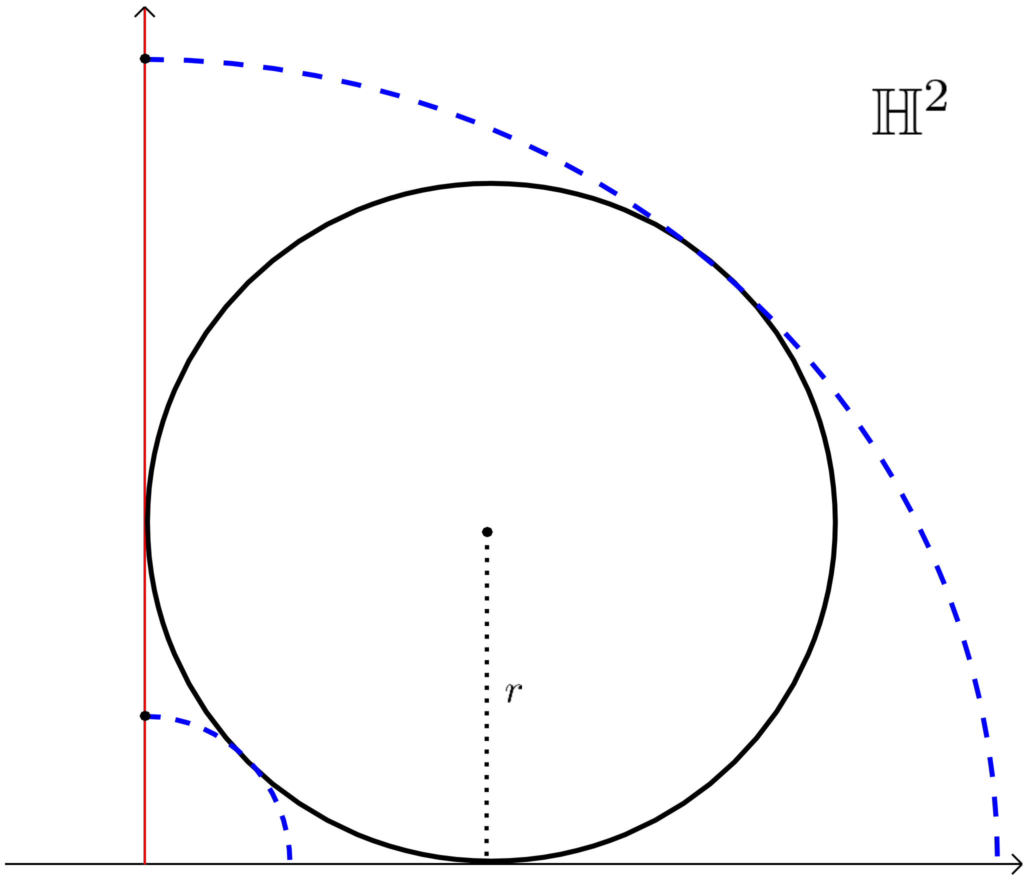
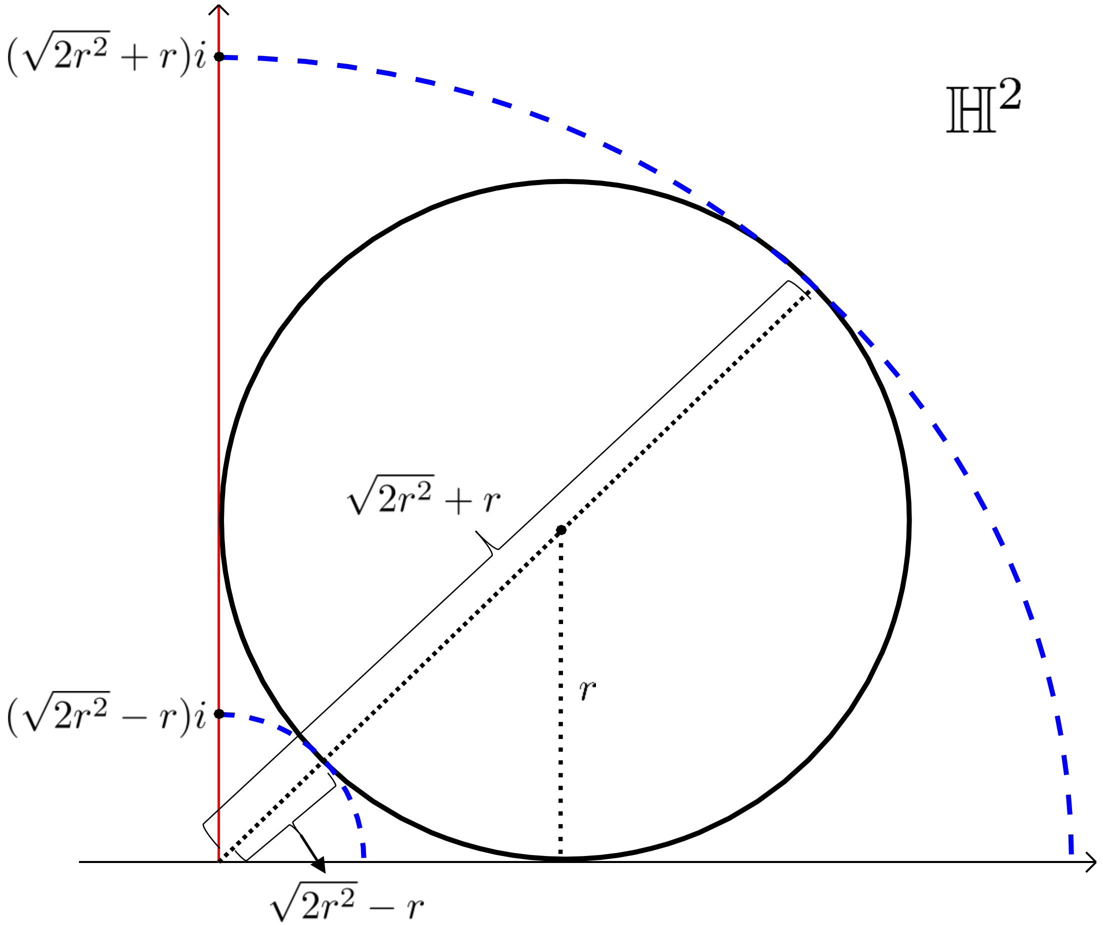


Contracting property

Every geodesic of $\mathbb{H}^2$ is strongly contracting (Arzhantseva-Cashen-Gruber-Hume).

$$d_{\mathbb{H}^2}((\sqrt{2r^2}+r)i, (\sqrt{2r^2}-r)i) = \log\left(\frac{\sqrt{2r^2}+r}{\sqrt{2r^2}-r}\right)$$
$$= \log(3+2\sqrt{2})$$



Quasi-axes

Let ϕ be pseudo-Anosov.

A quasi-axis is a geodesic preserved by some power of ϕ.

Proposition: Every pseudo-Anosov has a quasi-axis in $\text{MCG}(S)$.

Examples of contracting property

Let ϕ be pseudo-Anosov.

- Minsky: geodesic axis in Teichmuller space
- Brock - Masur - Minsky: quasi-axis in pants graph
- Duchin - Rafi: quasi-axis in MCG(S)

Main Theorem

Theorem (Rafi–V.)

a_ϕ

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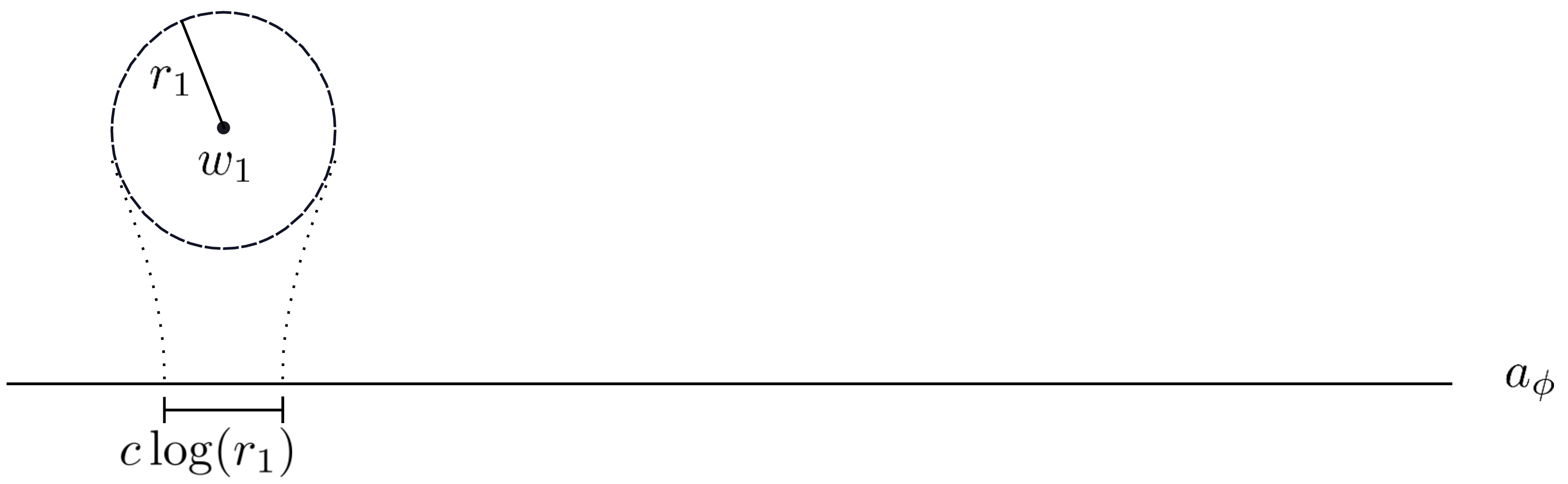
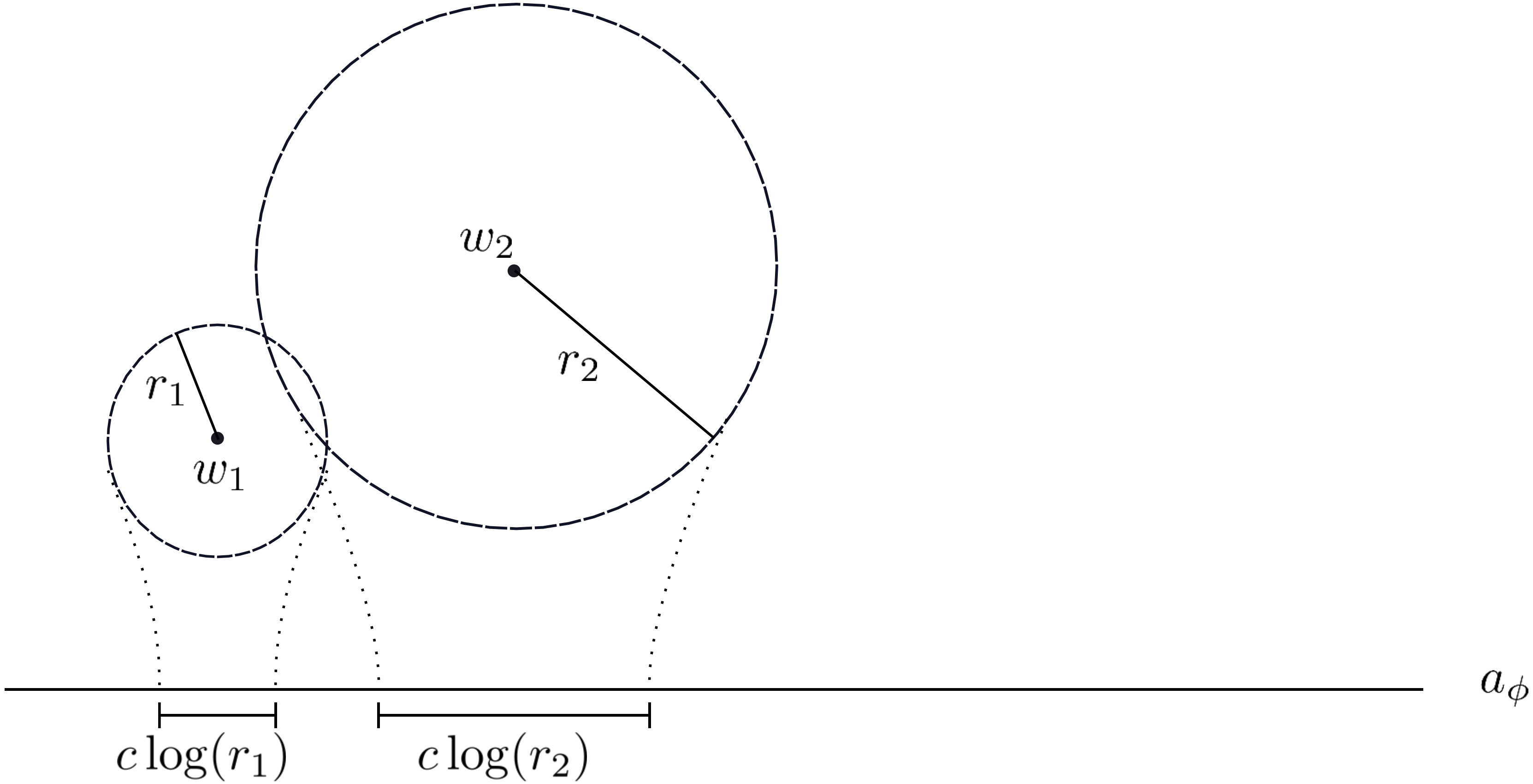


Theorem (Rafi–V.)



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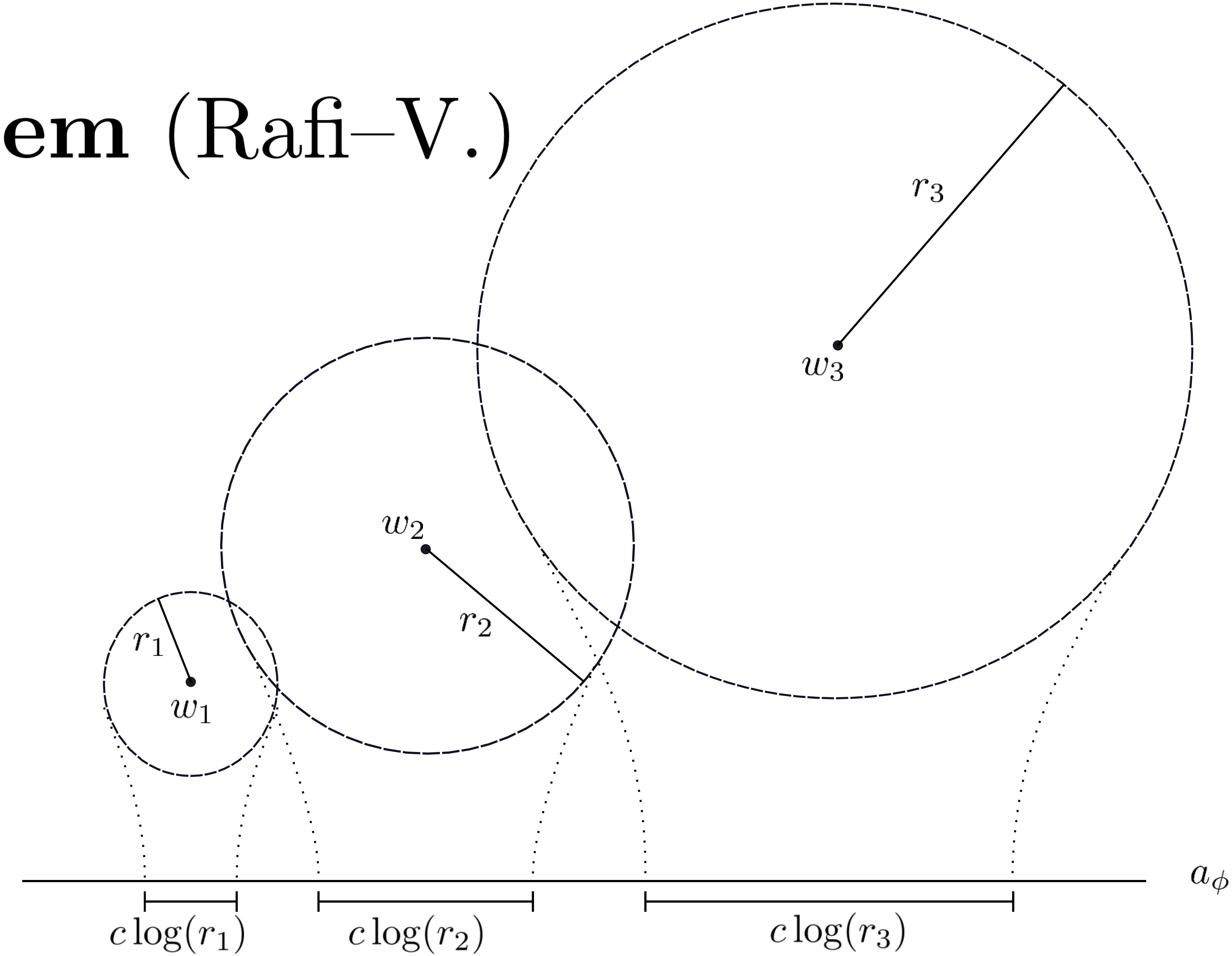


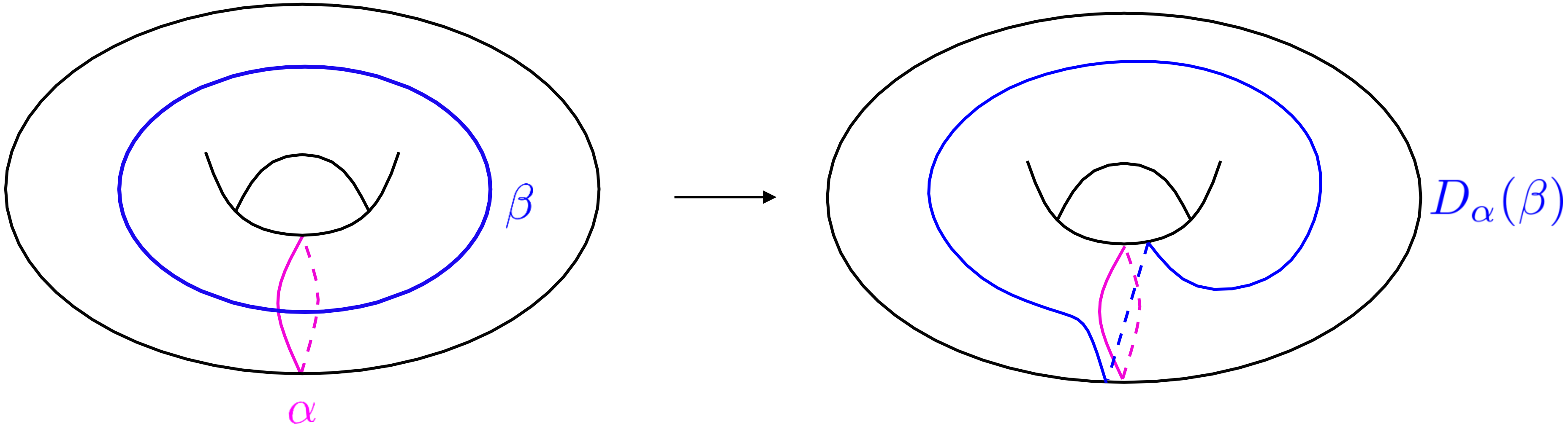
Main Theorem (Rafi - V.)

There exists a pseudo-Anosov $\phi \in \mathrm{PMCG}(S_{0,5})$ whose quasi-axis is not strongly contracting.

Proof Sketch

Generating mapping class groups

Dehn-Lickorish: finitely generated by Dehn twists, D_α

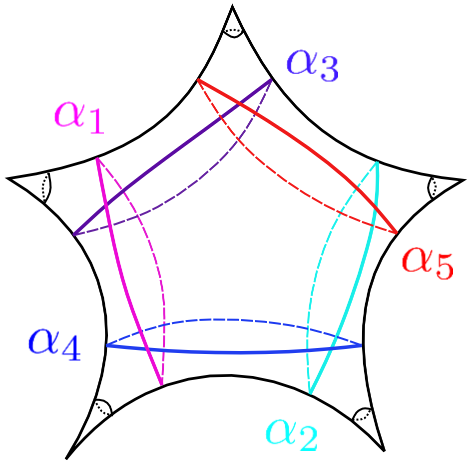


Positive twist: twist in right direction

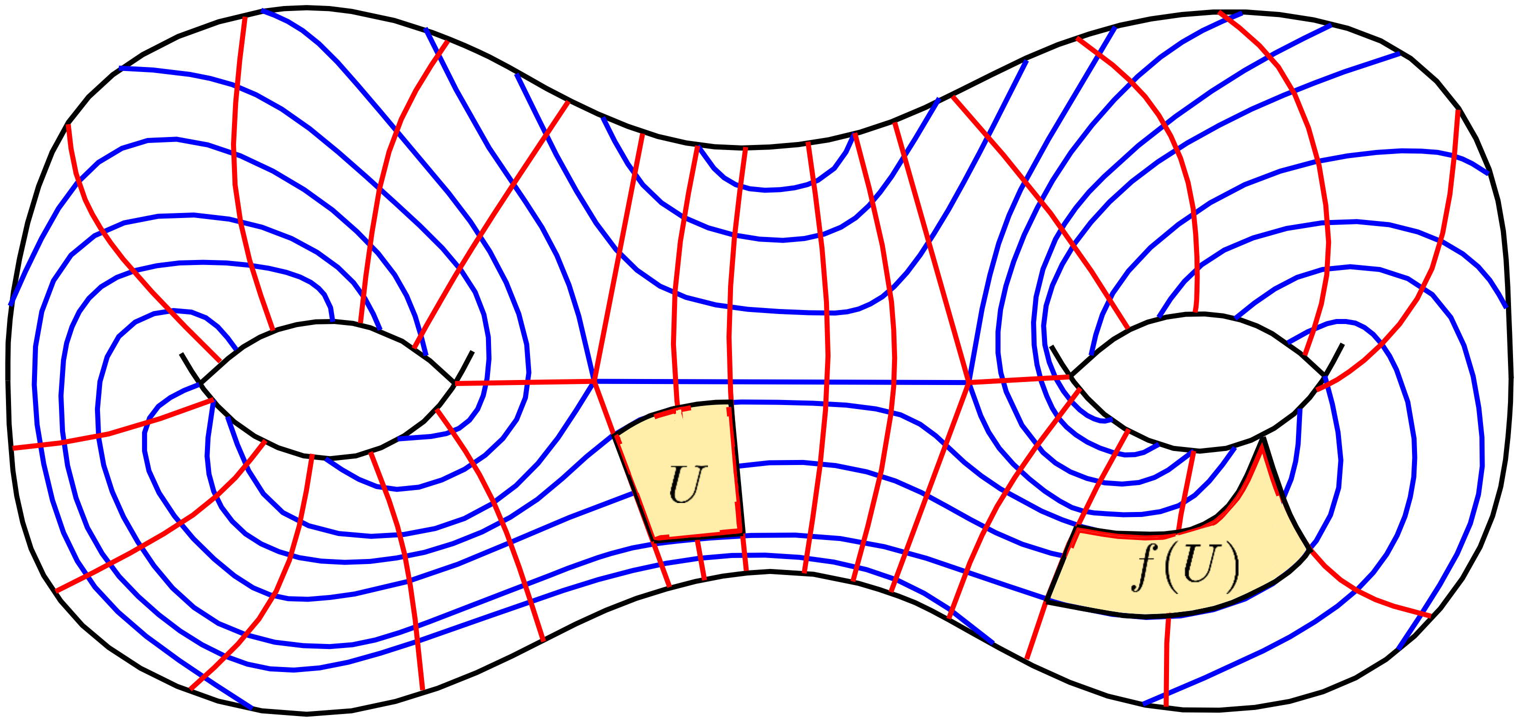
Set-up

Consider $S_{0,5}$ and let $n \gg 1$. Let the generating set of PMCG$(S_{0,5})$ be

$$\mathcal{S}_n = \{D_{\alpha_i}, D_{\alpha_i}^n D_{\alpha_j}^{-1} : i, j \in \mathbb{Z}_5,\ |i-j| = 1 \mod 5\}.$$



Pseudo-Anosov



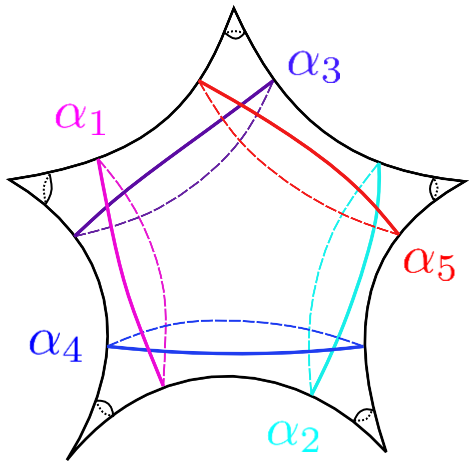
f maps no curve back to itself

Theorem (Thurston):
$\exists$ a number $\lambda > 1$ and a pair of foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ such that $f(\mathscr{F}^u) = \lambda \mathscr{F}^u$ and $f(\mathscr{F}^s) = \lambda^{-1} \mathscr{F}^s$.

Theorem (Rafi - V.)

$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1} \in \mathrm{PMCG}(S_{0,5})$$

is a pseudo-Anosov.



Aside: Generalization of construction

Algebraic invariants are used to determine when pseudo-Anosov mapping classes are different.

Either $\mathbb{Q}(\lambda + \lambda^{-1})$ is totally real or not.

↳ Hubert-Lanneau: For Thurstons construction, the trace field always totally real.

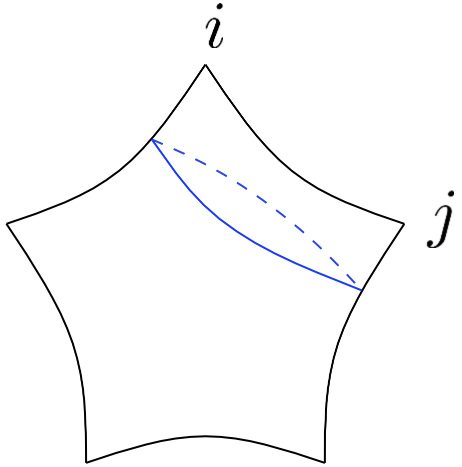
Either the Galois conjugates of λ are on the unit circle or are not.

↳ Shin-Strenner: For Penners construction, the Galois conjugates are never on unit circle.

Theorem (V.): New construction of pseudo-Anosov mapping classes which differ from both Penner and Thurston's construction.

Finding explicit geodesics

(i, j)–curve:



α consecutive if $|i - j| = 1 \mod 5$
α non-consecutive otherwise

Finding explicit geodesics (cont.)

Theorem (Rafi-V.): There exists a homomorphism $h : \mathrm{PMCG}(S_{0,5}) \longrightarrow \mathbb{Z}$ where

$$h(D_\alpha) = 1 \qquad \text{if } \alpha \text{ consecutive}$$
$$h(D_\alpha) = -1 \qquad \text{otherwise}$$

Question: What does this homomorphism mean geometrically?

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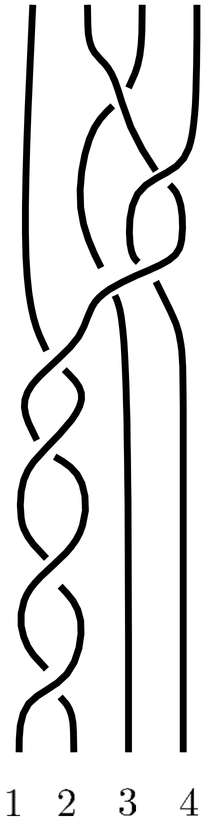
$h : \mathrm{PMCG}(S_{0,5}) \longrightarrow \mathbb{Z}$
$h(D_\alpha) = 1$ if α consecutive
$h(D_\alpha) = -1$ otherwise



$= B = D^1_{2,4} D^2_{1,2}$

$$\begin{aligned} h(B) &= h(D_{2,4} D^2_{1,2}) \\ &= h(D_{2,4}) + h(D_{1,2})^2 \\ &= (-1) + 2 = 1 \end{aligned}$$

Does h count number of braid crossings?

No. There are 8 braid crossings.

Does h count ordered crossings?

No. If we count a left over right crossing as $+1$, and a right over left crossing as -1, we get 6.

This problem is still open!

Finding explicit geodesics (cont.)

Theorem (Rafi-V.): There exists a homomorphism $h : \mathrm{PMCG}(S_{0,5}) \longrightarrow \mathbb{Z}$ where

$$h(D_\alpha) = 1 \qquad \text{if } \alpha \text{ consecutive}$$
$$h(D_\alpha) = -1 \qquad \text{otherwise}$$

Lemma: h gives a lower bound on the word length of any element in $\mathrm{PMCG}(S_{0,5})$.

Example

Consider $f = D_{\alpha_1}^{n^2-1} \in \mathrm{PMCG}(S_{0,5})$

Notice: $h(D_{\alpha_1}^{n} D_{\alpha_2}^{-1}) = n - 1$

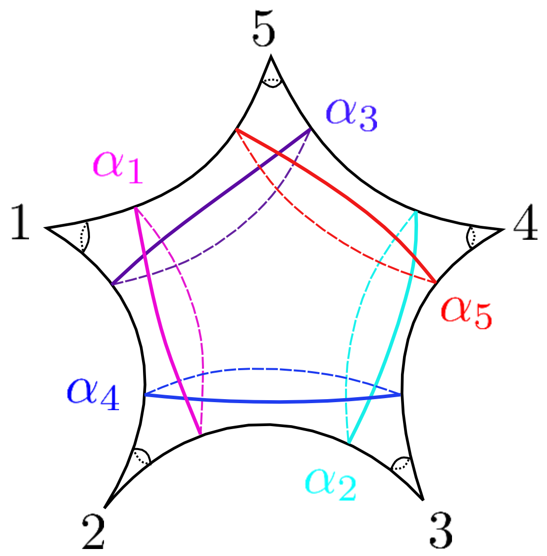
$$h(f) = n^2 - 1 = (n-1)\underbrace{(n+1)}_{\text{Lower bound of } \|f\|}$$

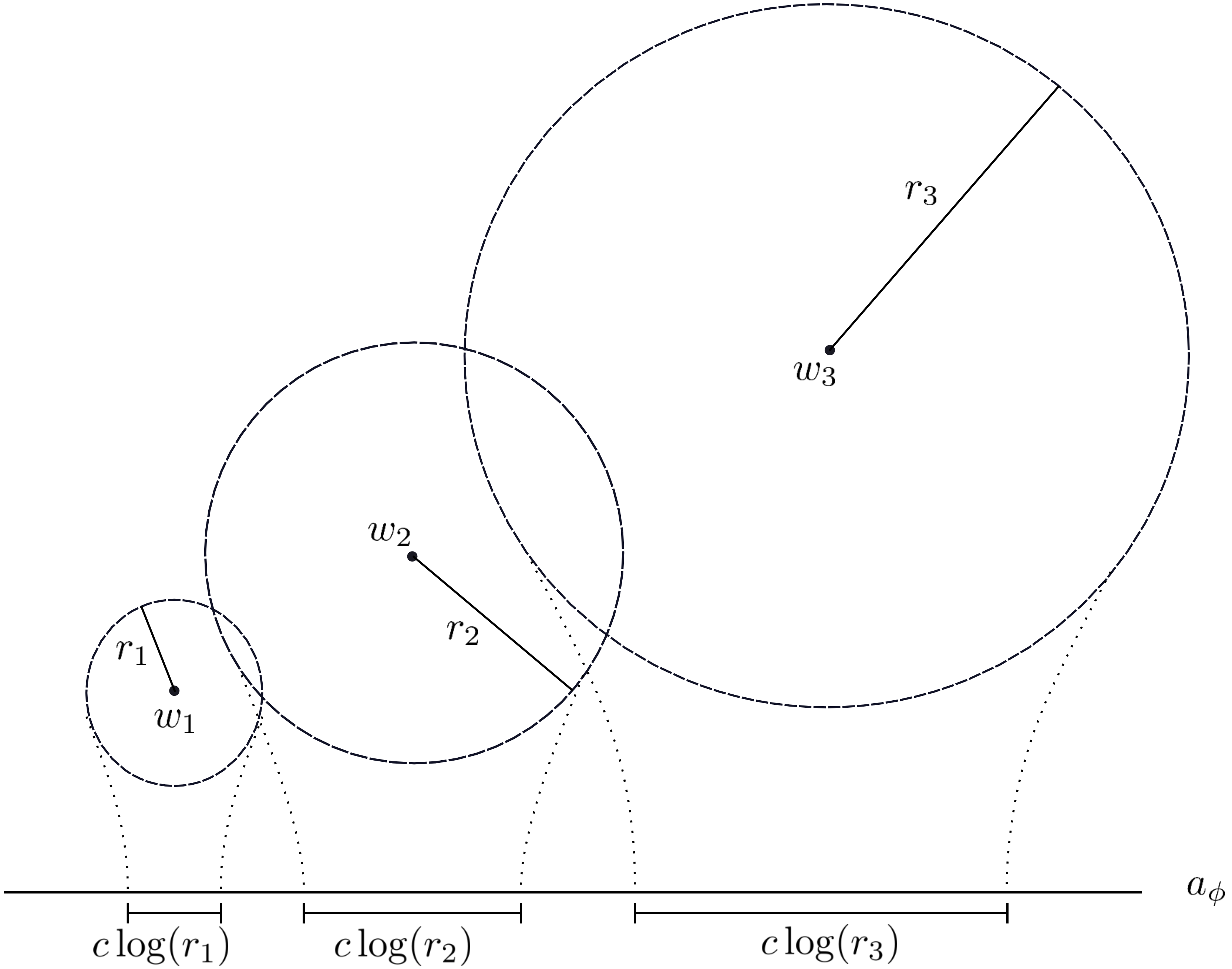
$$f = (D_{\alpha_1}^{n} D_{\alpha_2}^{-1})^{n}\ (D_{\alpha_2}^{n} D_{\alpha_1}^{-1})^{1}$$

$$= D_{\alpha_1}^{n^2} D_{\alpha_2}^{-n} D_{\alpha_2}^{n} D_{\alpha_1}^{-1}$$

Note: D_{α_1} and D_{α_2} commute

$$= D_{\alpha_1}^{n^2-1}$$



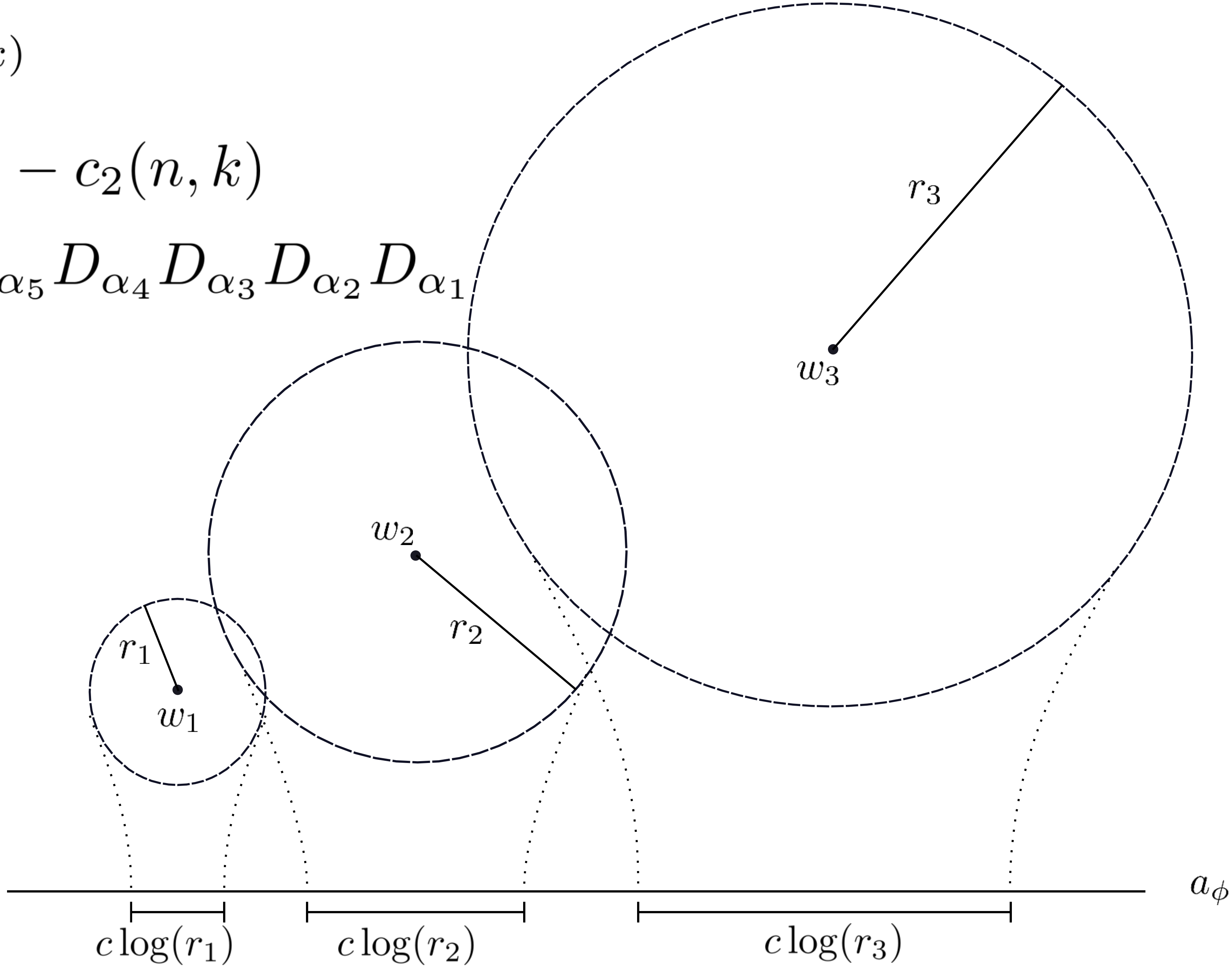


r_3
w_3
w_2
r_2
r_1
w_1
a_ϕ
$c\log(r_1)$
$c\log(r_2)$
$c\log(r_3)$

Sequence: $w_k = D_{\alpha_1}^{c_1(n,k)}$

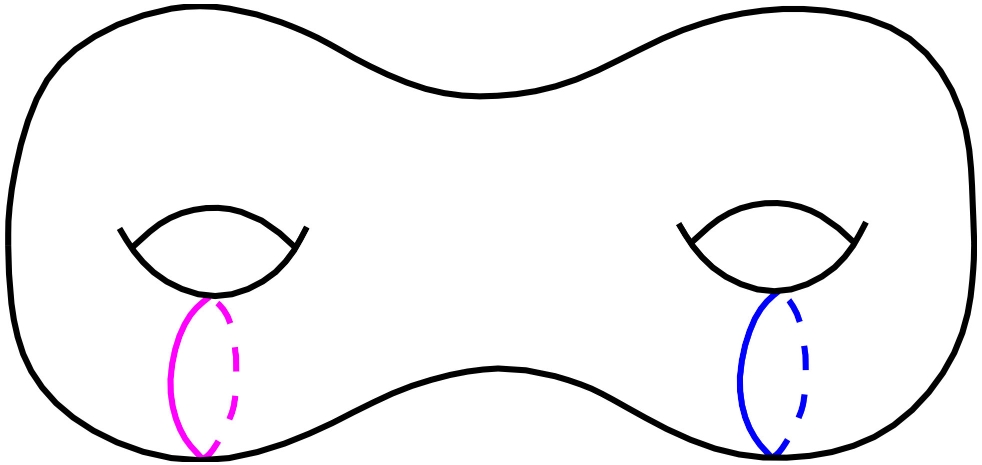
Radii: $r_k = d(w_k, \phi^{k/5}) - c_2(n,k)$

Pseudo-Anosov: $\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$



Bounding projections in MCG(S) is hard!

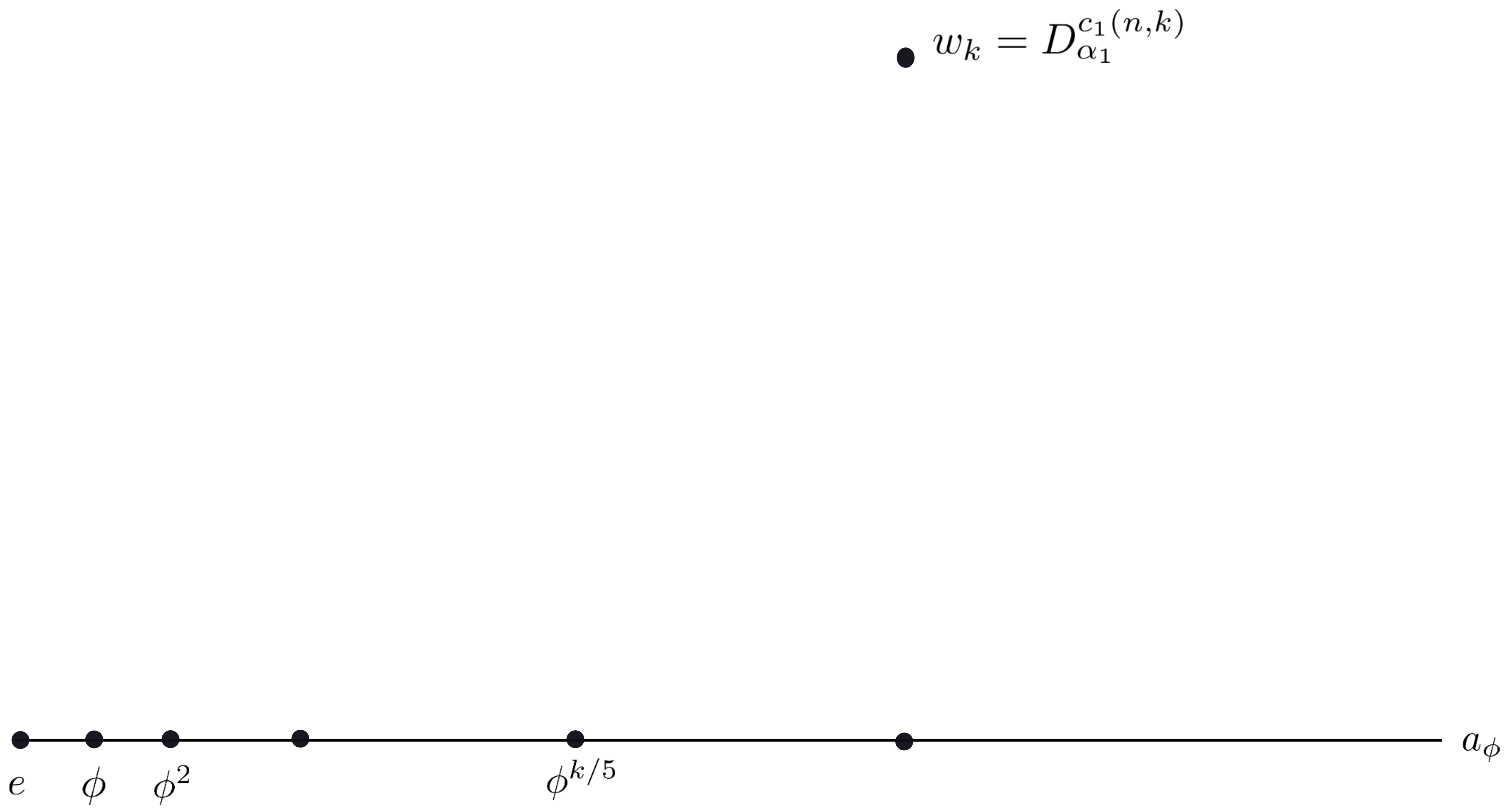
There are both flat and hyperbolic directions.



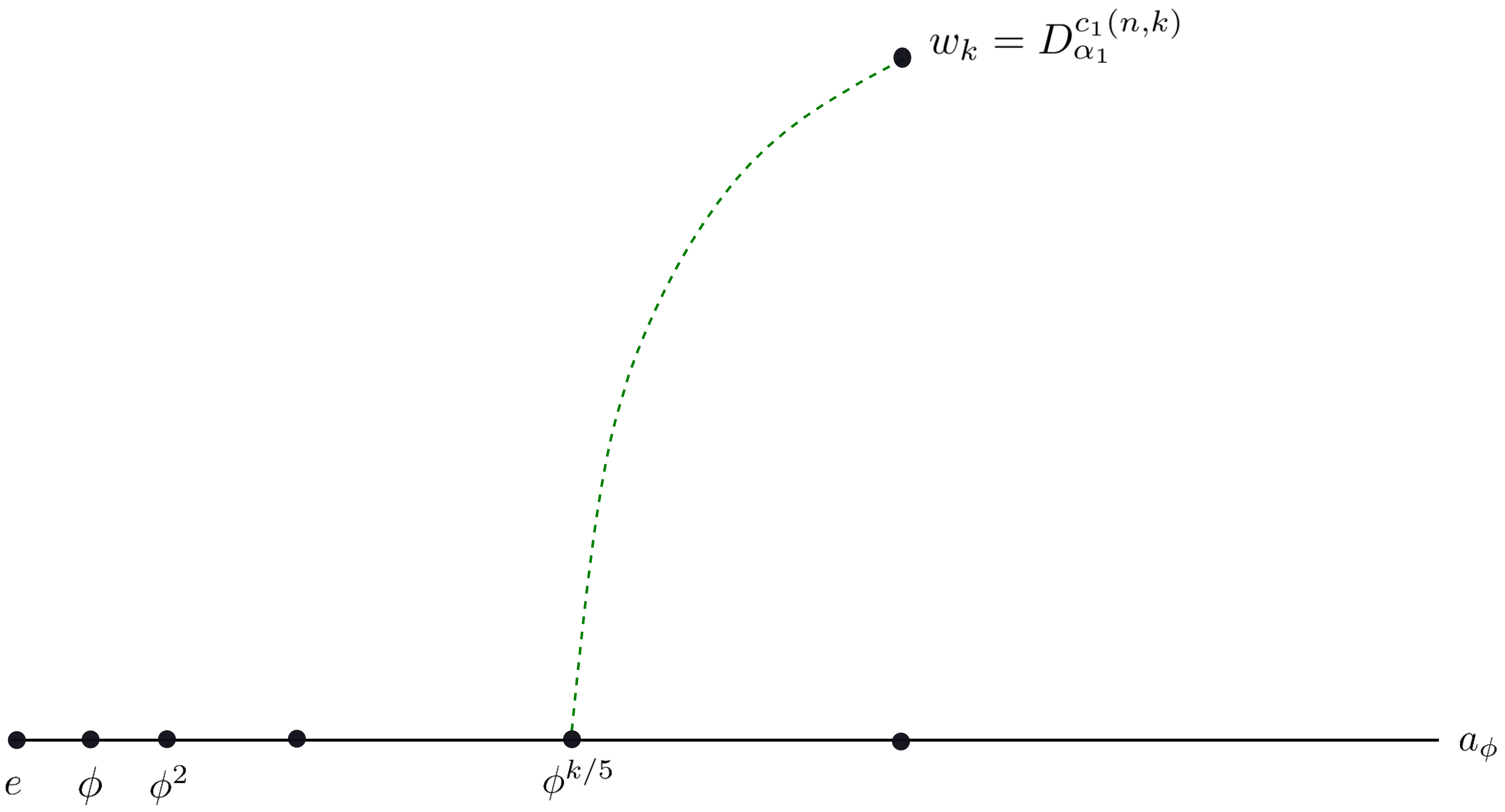
Dehn twists about disjoint curves
$\rightsquigarrow$ flats in Cayley graph

Duchin-Rafi: quasi axis for pseudo-Anosov in MCG(S) has the contracting property
$\rightsquigarrow$ hyperbolic direction in Cayley graph

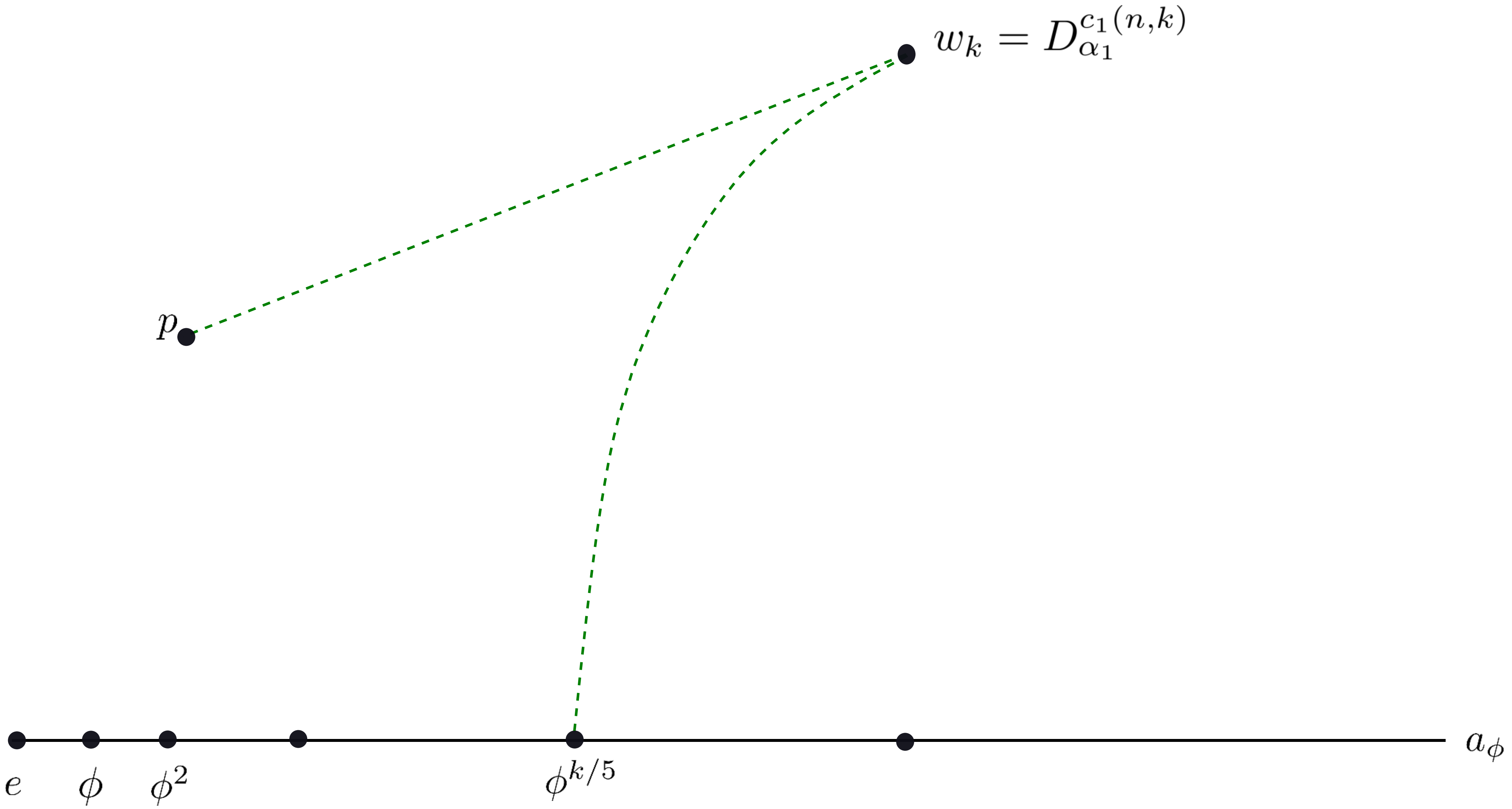
$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$$



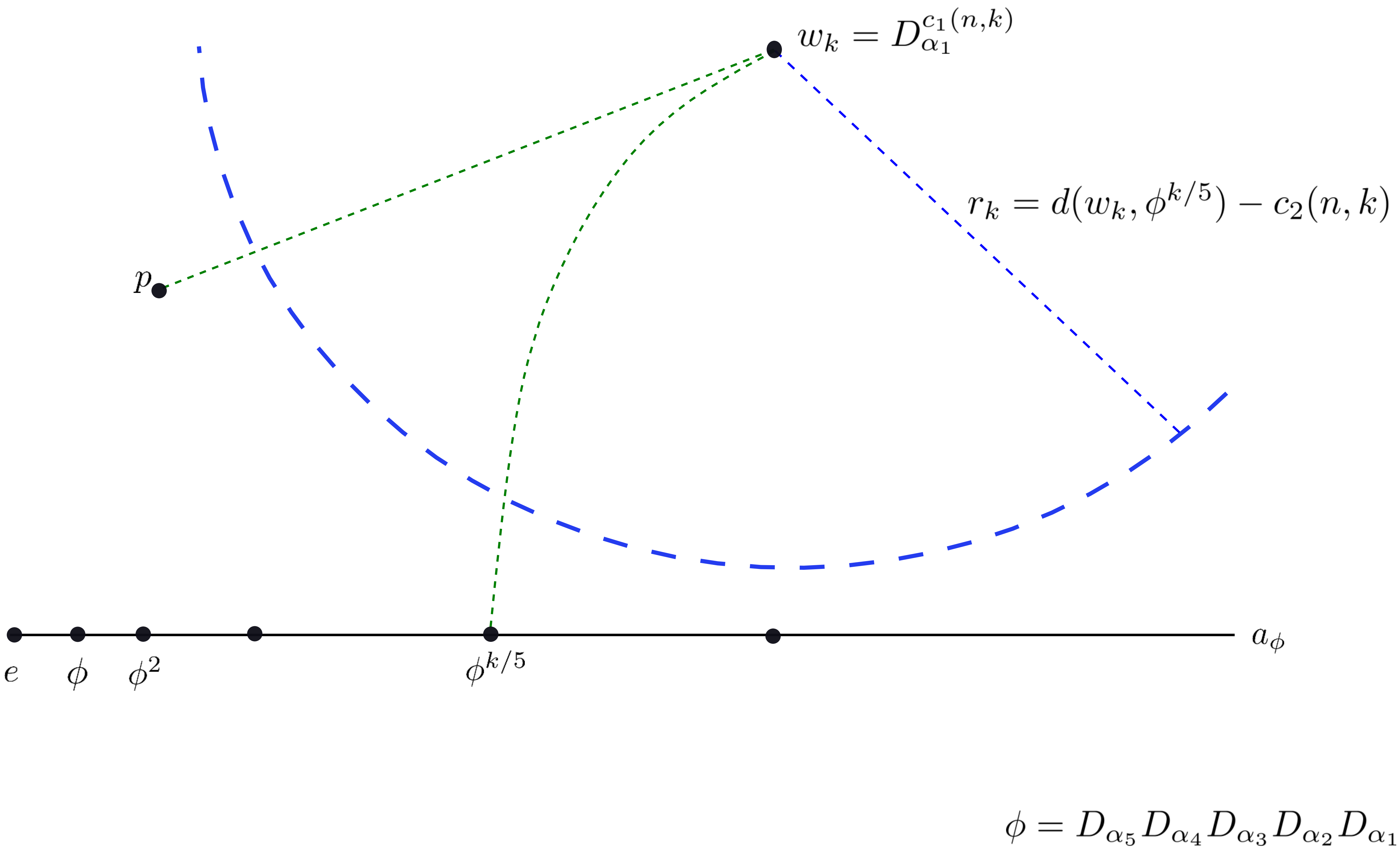
$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$$



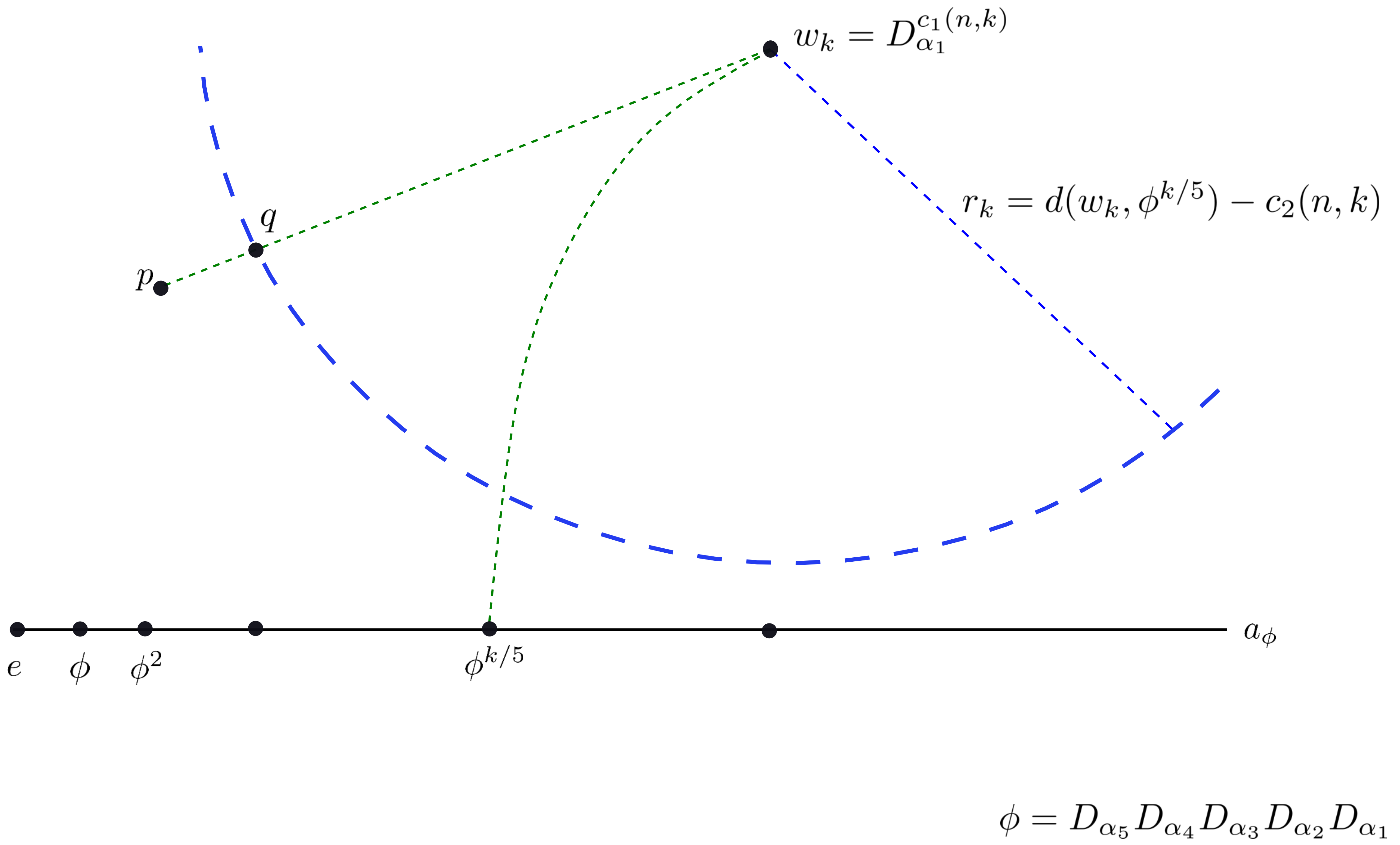
$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$

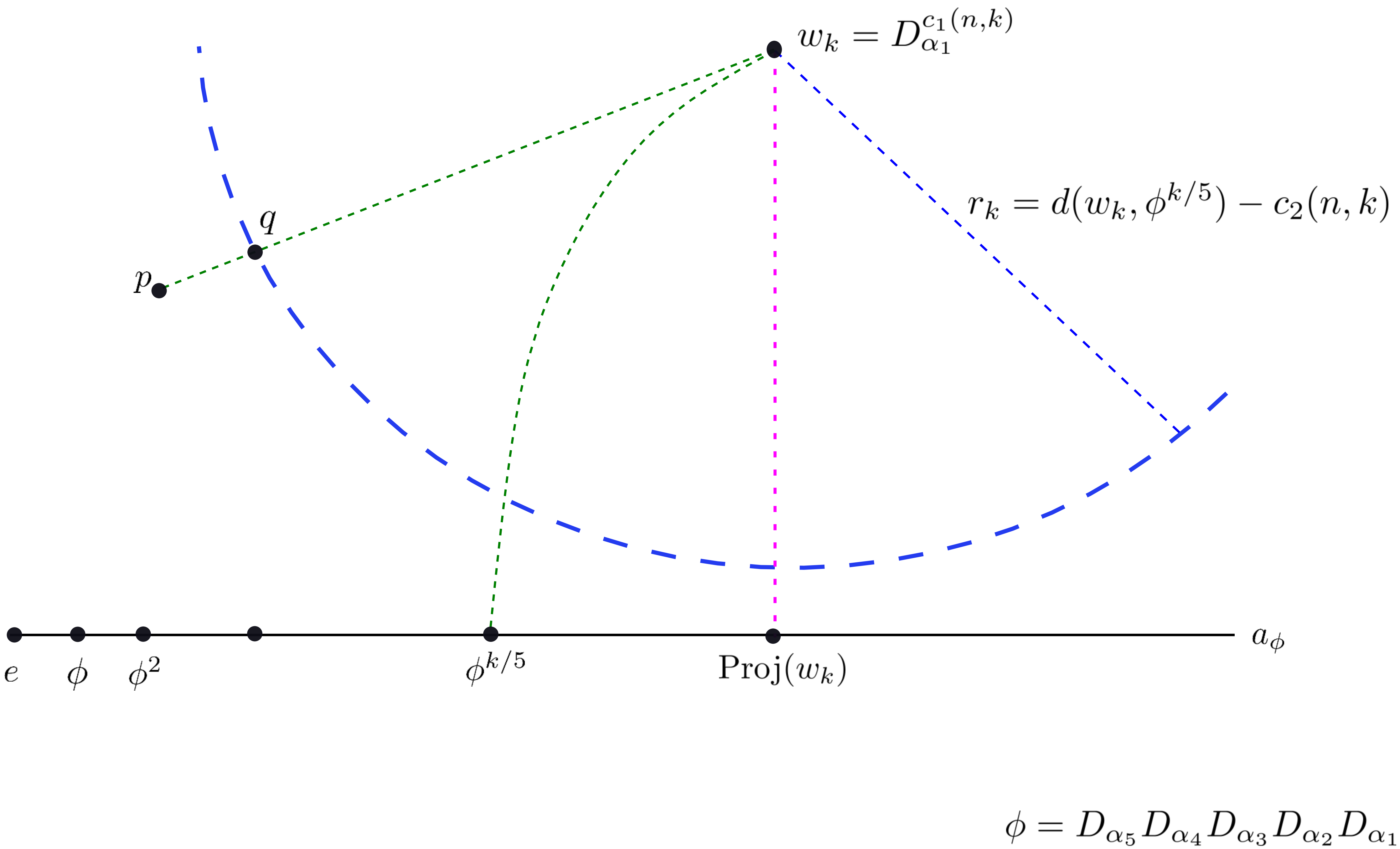


$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$$

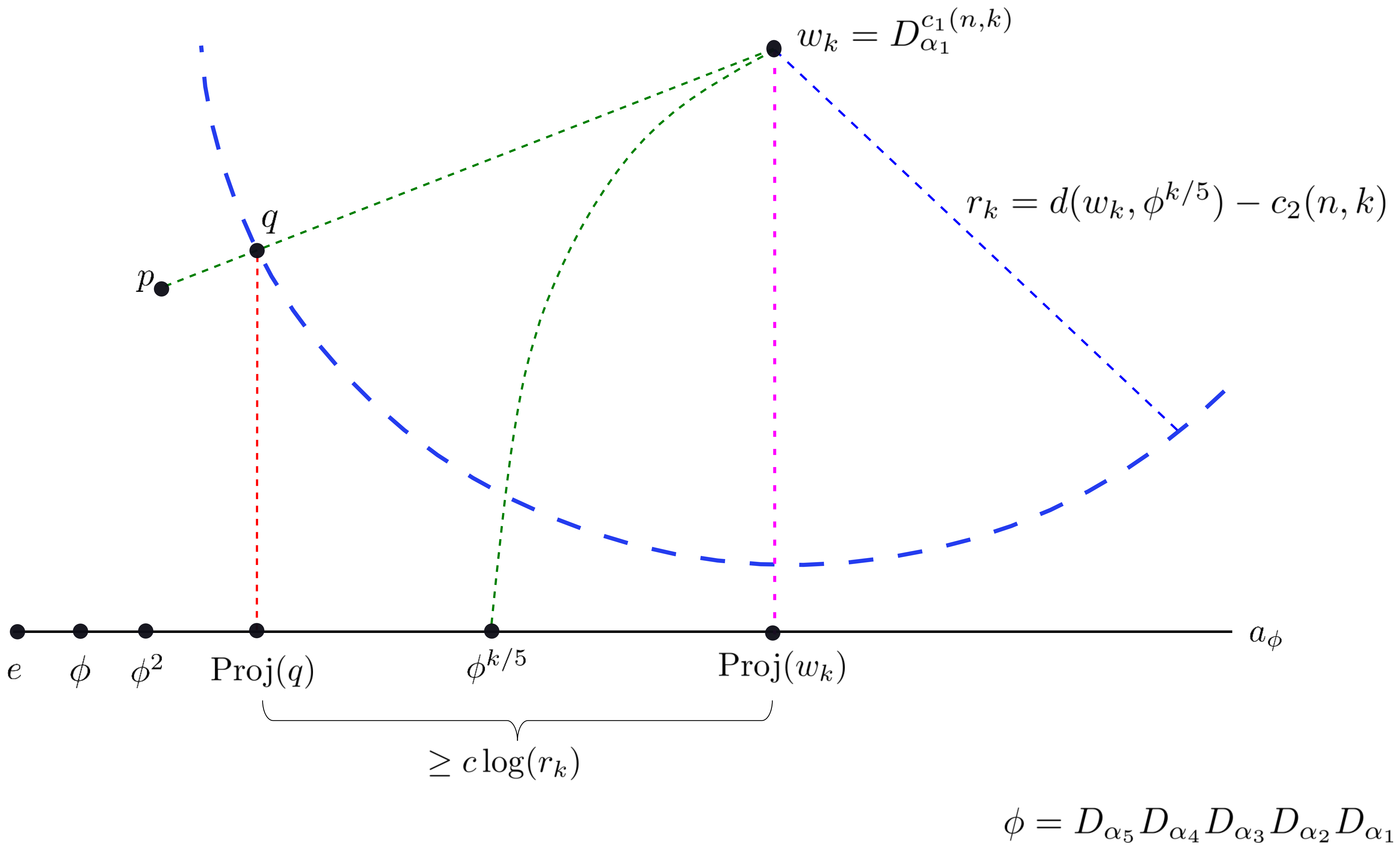


$w_k = D_{\alpha_1}^{c_1(n,k)}$
$r_k = d(w_k, \phi^{k/5}) - c_2(n,k)$
p
e
ϕ
ϕ^2
$\phi^{k/5}$
a_ϕ
$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$


$w_k = D_{\alpha_1}^{c_1(n,k)}$
$r_k = d(w_k, \phi^{k/5}) - c_2(n,k)$
q
p
a_ϕ
e
ϕ
ϕ^2
$\phi^{k/5}$
$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$



$w_k = D_{\alpha_1}^{c_1(n,k)}$
$r_k = d(w_k, \phi^{k/5}) - c_2(n,k)$
q
p
a_ϕ
e
ϕ
ϕ^2
$\phi^{k/5}$
$\mathrm{Proj}(w_k)$
$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$


$w_k = D_{\alpha_1}^{c_1(n,k)}$
$r_k = d(w_k, \phi^{k/5}) - c_2(n,k)$
q
p
a_ϕ
e
ϕ
ϕ^2
$\mathrm{Proj}(q)$
$\phi^{k/5}$
$\mathrm{Proj}(w_k)$
$\geq c\log(r_k)$
$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}$

Open problems

- Does there exist a pseudo-Anosov whose quasi-axis is strongly contracting?
- (Farb's Conjecture) Is the set of pseudo-Anosov elements generic with respect to the word metric?
- Are there pseudo-Anosov elements whose quasi-axis is not strongly contracting on surfaces with positive genus?
- Can the quasi-axis of ϕ be strongly contracting with respect to one generating set, but not with repect to another?